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The Gains from Diversification Reconsidered: Transaction Costs and Superior Information

BY AZRIEL LEVY AND MILES LIVINGSTON

While theory implies broad diversification, many investors hold portfolios with a surprisingly small number of securities. This paper shows that transactions costs and superior information may provide strong incentives not to diversify.

I. INTRODUCTION

In the years since Markowitz's seminal article (1952) and book (1959), much has been written about the benefits of portfolio diversification. While theory implies broad diversification, many investors hold portfolios with a surprisingly small number of securities.[1] This paper examines three explanations for this lack of diversification in a mean-variance world with identical means and with a risk-free asset.[2] First, most of the benefits of diversification are shown theoretically to occur for relatively small numbers of securities, providing formal proof of the simulation results of Evans and Archer (1968). Second, transactions costs are shown to significantly reduce the optimal number of securities in the portfolio. With transactions costs, portfolio separation is shown to breakdown, in general; investors with different wealth levels, as well as investors with the same wealth levels, typically hold different optimal portfolios. Third, superior information about a particular security is found to provide strong incentives to concentrate a portfolio and reduce the optimal number of securities in the portfolio. These three factors combine to imply optimal portfolios with relatively small numbers of securities. In this world of concentrated portfolios, the Generalized CAPM of Levy (1978) provides an explanation of equilibrium.

The first part of the paper examines the benefits and costs of diversification assuming a mean-variance framework with a risk-free asset and identical return distributions for all securities (i.e., equal means and standard deviations). There are diversification benefits if the correlation coefficients between the pairs of securities are less than 1.0. As securities are added to the portfolio, the risks are reduced

[1] See Blume et al. (1974), Blume and Friend (1975, 1978) and Lease et al. (1976) for evidence that many investors hold small numbers of securities in their portfolios. French and Poterba (1991) provide evidence that international investors hold portfolios over-concentrated in domestic securities.

[2] The mean-variance framework applies if return distributors are normal and/or if quadratic utility functions apply. Levy and Markowitz show that the mean variance optimization is a good approximation for the optimal expected utility for a wide range of utility functions and empirical distributions. See Levy and Markowitz (1979), Kroll, Levy and Markowitz (1984), Pulley (1981, 1983, 1985).

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at a decreasing rate. For correlation coefficients of .50 or larger, close to 90% of the maximum diversification benefits are shown to occur with a portfolio size of 10 securities, confirming the simulation results of Evans and Archer (1968). If the standard deviations differ so that the largest standard deviation is no more than $1 + k$ times the average standard deviation, the portfolio size required for a particular level of diversification is no more than $1 + k$ times as large as the case with identical standard deviations.

The second part of the paper adds transaction costs. The optimal portfolio size is shown to depend upon investor wealth, the form of the cost function and investor utility functions. That is, portfolio separation does not hold, in general. For a given wealth level, portfolio composition is shown to depend upon investor utility functions except in special cases. If transactions cost per dollar of wealth decrease as wealth increases, portfolio size is shown to increase as wealth increases. Then, small investors will hold a small number of securities in their portfolio, consistent with the empirical findings of Blume et alia (1974, 1975, 1978) and Lease et alia (1976). Small investors will tend to find mutual funds more cost effective than direct investment.

Optimal portfolio size is examined for brokerage cost functions currently available in the market. Large volume discount brokers charge a fixed fee per share. This per share charge causes a downward parallel shift in the efficient frontier and implies highly diversified portfolios for the institutions using this type of broker. Small volume discount brokers have (piecewise) linear brokerage cost functions. This type of brokerage cost function sharply reduces the benefits of diversification and implies relatively small portfolios for the small investors who would use these small volume discount brokers.

The third part of the paper adds a superior asset. The existence of superior information about a specific security (or portfolio of securities) is shown to provide a strong incentive to concentrate the portfolio. Surprisingly small amounts of superior information are shown to bring about considerable portfolio concentration.[3]

The last part of the paper considers transactions costs and a superior asset. The introduction of a superior asset reduces the number of average securities in the portfolio, in general. Thus, transactions costs and superior information act in the same direction, tending to reduce the number of securities held in a portfolio.

II. PORTFOLIO SIZE WITH AVERAGE SECURITIES

This section examines the impact of diversification upon portfolio standard deviation in a mean-variance world. Initially, there are assumed to be N distinct, risky securities, each of which has the same expected return of u_a and standard deviation

[3]These results are consistent with the empirical results of Blume & Friend (1975, 1978), Lease et alia (1976) and French and Porterba (1991) who find that international investors may over concentrate their portfolios in domestic securities because of superior information. The results are also consistent with the theoretical and simulation results of Best and Grauer (1991).

of σ_a. The correlation coefficient of returns is r, the same for all securities. Under these assumptions, we show that (1) the percentage reduction in portfolio standard deviation is independent of the level of standard deviation, (2) the percentage reduction increases at a decreasing rate as portfolio size increases, (3) a reduction in the correlation coefficient increases the percentage reduction of portfolio standard deviation, (4) a very large proportion of the attainable portfolio diversification tends to occur by the time portfolio size is ten securities.

The later part of this section suspends the assumptions of the same standard deviation and correlation coefficient for all securities. The identical results apply for the average variance and average covariance of securities. This finding explains the simulation results of Evans and Archer.

THE SAME VARIANCE AND COVARIANCE FOR ALL SECURITIES

When the standard deviation is σ_a for all securities and the correlation coefficient of returns is r for all securities, the portfolio standard deviation for an equally-weighted portfolio of n securities denoted by σ_n is

$$\sigma_n = \sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sigma_a^2 + \frac{2}{n^2}\sum_{i=1}^{n}\sum_{i\neq j}^{n}\sigma_a^2 r} \tag{1}$$

Note that the term $\sigma_a^2 r$ is the covariance between the pairs. By taking the sums, equation (1) simplifies to

$$\sigma_n = \sqrt{\frac{\sigma_a^2}{n} + \frac{(n-1)\sigma_a^2 r}{n}} \tag{2}$$

$$\sigma_n = \sqrt{\sigma_a^2 r + \frac{\sigma_a^2 - \sigma_a^2 r}{n}} = \sigma_a\sqrt{r + \frac{1-r}{n}} = \sigma_a\sqrt{r^*} \tag{3}$$

Where $r^* = r + (1-r)/n$. That is, the portfolio standard deviation is the square root of the covariance plus one nth of the difference between the variance and covariance.[4]

Figure 1 graphically shows the impact of increased portfolio size on the portfolio standard deviation.[5] For a single-security portfolio, the portfolio standard deviation is σ_a. For a 2-security portfolio, the portfolio standard deviation is σ_2, which is smaller than σ_a. For a 3-security case, the portfolio standard deviation is σ_3, which is less than σ_2. As securities are added, the risk is reduced. But, the risk reduction gets smaller and smaller as securities are added to the portfolio. Thus, the marginal benefit from diversification decreases as portfolio size increases.

[4] Markowitz (1959) has shown essentially the same result. Mao (1970) shows this result.

[5] Ben Horim and Levy (1980) show that the standard deviation is a better measure of risk than the variance.

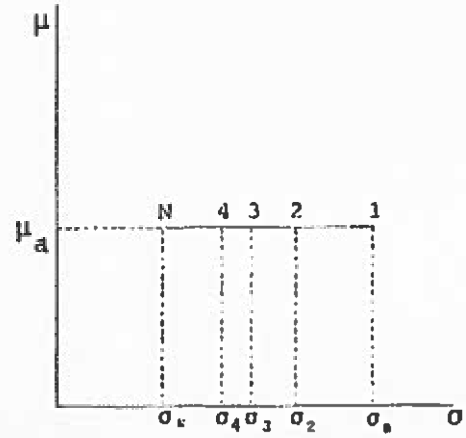

Figure 1: Portfolio Size and Portfolio Standard Deviation

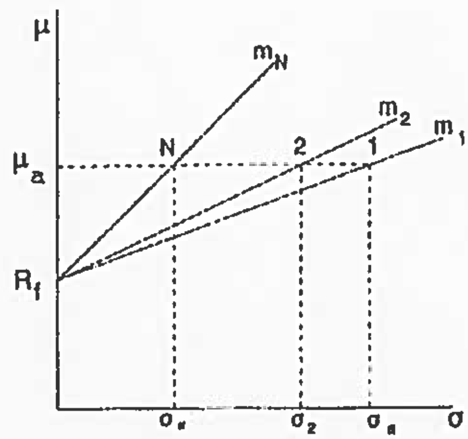

Figure 2: Portfolio Size and Portfolio Standard Deviation

The term $(\sigma_a^2 - \sigma_a^2 r)/n$ in equation (3) is the diversifiable risk in a portfolio. As n gets large, the diversifiable risk gets quite small and the portfolio standard deviation approaches the square root of the covariance. A portfolio with an infinite number of securities has a standard deviation of $\sigma_a(r)^{.5}$.

If investors can borrow or lend at the risk-free interest rate, the possible portfolios are shown in Figure 2. Along the line m_1, the investor holds a combination of the risk-free asset and a portfolio of one risky security. Along the line m_2, the investor holds a combination of the risk-free security and a portfolio of two risky securities. And so on.

Risk averse investors in a mean-variance world want to maximize their excess return per unit of risk, namely

$$\text{Maximize} \frac{u_n - R_f}{\sigma_n} = \text{Maximize} \frac{u_a - R_f}{\sigma_a\sqrt{r + \frac{1-r}{n}}} \tag{4}$$

Geometrically, the optimal portfolio will lie on the line m, a combination of the risk-free asset and a portfolio of the entire universe of N risky assets. This portfolio is a fully diversified or minimum standard deviation portfolio.

To see the relative benefit of adding securities to the portfolio, consider the percent reduction in portfolio standard deviation between a single security with standard deviation of σ_a and an n-security portfolio

$$\% \text{ reduction} = \frac{\sigma_a - \sigma_n}{\sigma_a} = 1 - \frac{\sigma_n}{\sigma_a} = 1 - \sqrt{r + \frac{1-r}{n}} \tag{5}$$

Equation (5) indicates several interesting things. First, the percentage reduction in portfolio standard deviation is independent of the level of standard deviation. Second, the percentage reduction increases at a decreasing rate as portfolio size n increases.[6] As n approaches infinity, the percentage reduction approaches $1-(r)^{.5}$. Third, as the correlation coefficient decreases, the percentage reduction increases for a given increment in portfolio size.[7]

The relative magnitude of the diversification benefit is clearer from examining the percentage reduction relative to the maximum attainable if an infinite number of assets is included in the portfolio.

$$\text{Percent of attainable risk reduction} = \frac{\%\text{ reduction for finite } n}{\%\text{ reduction for infinite } n}$$

$$x = 1 - \frac{\sqrt{r + \frac{1-r}{n}}}{1 - \sqrt{r}} \tag{6}$$

This measure adjusts for the fact that the maximum attainable diversification depends upon the correlation coefficient r. If r equals 1, the percent of attainable risk reduction equals 1 for all values of n.

Solve equation (6) for n, the number of securities necessary to attain a level of attainable risk reduction x.[8]

$$n = \frac{1-r}{[1 - x(1-\sqrt{r})]^2 - r} \tag{7}$$

[6] $d\,(\%\text{ reduction})/dn > 0$ and $d^2\,(\%\text{ reduction})/dn^2 < 0$.

[7] $d\,(\%\text{ reduction})/dr < 0$.

[8] Mao (1970) examines the percentage reduction in the unit price of risk and derives a somewhat different formula.

The case of a correlation coefficient of zero provides a simple analytical result. For $r = 0$

$$n = \frac{1}{(1-x)^2} \tag{8}$$

From equations (7) and (8), higher levels of risk reduction require increasingly higher numbers of securities. Table 1 shows the number of securities required for a given level of risk reduction for several levels of the correlation coefficient.

Several things are apparent from Table 1. First, a large proportion of the attainable risk reduction occurs by the time portfolio size is 10 securities. Second, as the correlation coefficient increases, the portfolio size for attaining a given level of risk reduction gets smaller. However, for correlation coefficients of .50 or higher, changes in the correlation coefficient have relatively little impact upon the percent of attainable risk reduction. For $r \geq .50$, 88% of attainable risk reduction occurs with 10 securities.

In several situations, portfolio indexes are constructed to measure performance. These include country indexes and industry indexes. Table 1 provides evidence that the index standard deviation will be fairly close to the true standard deviation even though only 8 or 10 securities compose the index, as long as $r \geq .50$.

Different Variances and Covariances

The preceding results assumed that the variances and covariances for all securities are the same. In the case where the variances and covariances differ, the identical results apply for the average variance and average covariance. Specifically, equations (6)–(8) and Table 1 are the same with the correlation coefficient r being replaced by the ratio (average covariance)/(average variance).

The case of the average variance and average covariance provides an explanation of the simulation results of Evans and Archer (1968) and others. By simulating diversification benefits using 60 or more portfolios of varying sizes, these researchers are effectively showing diversification for the average variance and average covariance. Evans and Archer (1968) have concluded that a portfolio size of 8 or 10 securities is desirable since most diversification benefits are already achieved. This is exactly what Table 1 shows for values of the (average covariance)/(average variance) of .50 or higher. If this ratio is close to zero, substantially larger portfolios are required to achieve diversification results.

Upper Bounds on the Number of Securities

If the variances differ, Table 1 understates the number of securities required to be confident that an individual portfolio will achieve diversification benefits. Appendix A shows that no more than twice as many securities are required to achieve

Table 1: Portfolio Size (n) to Achieve a Given Level of Risk Reduction

% Risk	Correlation Coefficient (r)						
Reduction (x)	0.00	0.25	0.50	0.60	0.70	0.80	0.90
0.100	1.2	1.1	1.1	1.1	1.1	1.1	1.1
0.200	1.6	1.3	1.3	1.3	1.3	1.3	1.3
0.300	2.0	1.6	1.5	1.5	1.5	1.5	1.4
0.400	2.8	1.9	1.8	1.8	1.7	1.7	1.7
0.500	4.0	2.4	2.2	2.1	2.1	2.1	2.0
0.600	6.3	3.1	2.8	2.7	2.6	2.6	2.5
0.700	11.1	4.3	3.8	3.7	3.6	3.5	3.4
0.800	25.0	6.8	5.8	5.6	5.4	5.2	5.1
0.820	30.9	7.6	6.5	6.2	6.0	5.8	5.7
0.840	39.1	8.7	7.3	7.0	6.8	6.6	6.4
0.860	51.0	10.0	8.4	8.0	7.7	7.5	7.3
0.880	69.4	11.8	9.8	9.4	9.0	8.8	8.5
0.900	100.0	14.3	11.8	11.3	10.9	10.5	10.2
0.920	156.2	18.0	14.8	14.2	13.6	13.2	12.8
0.940	277.8	24.3	19.9	18.9	18.2	17.6	17.1
0.960	625.0	36.8	29.9	28.5	27.3	26.4	25.6
0.980	2500.0	74.3	60.1	57.1	54.8	52.9	51.3
0.990	10000.0	149.3	120.5	114.4	109.7	105.8	102.7
0.992	15624.9	186.8	150.6	143.0	137.1	132.3	128.3
0.994	27777.4	249.3	200.9	190.7	182.8	176.5	171.1
0.996	62497.9	374.2	301.5	286.2	274.3	264.7	256.7
0.998	249976.7	749.2	603.2	572.5	548.8	529.5	513.4

$$\text{Portfolio Size} = \frac{1-r}{[1-x(1-\sqrt{r})]^2 - r} \tag{9}$$

diversification benefits (1) if the distribution of standard deviations is symmetric about its mean or (2) if the distribution of standard deviations is skewed so that the largest standard deviation is no bigger than twice the mean. In the general case where the distribution of standard deviations is nonsymmetric about the mean of the standard deviations, let the biggest standard deviation equal the average standard deviation times $1+k$. From Appendix A, the value of the optimal portfolio size cannot exceed the numbers in Table 1 times $1+k$. Thus, if the biggest standard deviation equals four times the average standard deviation (i.e., k equals four) and $r \geq .50$, no more than 49 (i.e., 5×9.8) securities are required to achieve 88% of the benefits of risk reduction.

III. AVERAGE SECURITIES AND TRANSACTION COSTS: THEORY

Without transaction costs, increased portfolio size has been shown to result in lower portfolio standard deviation. In a world with no transactions costs, the optimal portfolio is fully diversified, i.e., $1/N$ invested in each of the available N risky assets. With transactions costs, the diversification benefits must be balanced against the incremental cost of adding securities.

Portfolio Mean and Standard Deviation with Transactions Costs

Suppose an investor with an initial wealth of W invests in a portfolio of n risky assets. The total transactions costs are $D(n)$. These costs include brokerage fees, information costs, and the value of an investor's time. For simplicity, we assume that transactions costs are paid at the end of the investment period.

The cash flows on the investment at time zero and time one are

Time Zero:

$$-W = -\sum_{i=1}^{n} F_{i0} Z_i \tag{10}$$

Time One:

$$\sum_{i=1}^{n} F_{i1} Z_i - D(n) \tag{11}$$

where

F_{i0} = price of security i at time zero
F_{i1} = price plus dividends of security i at time one
Z_i = number of units of security i purchased

One plus the rate of return on the portfolio, R_p, is

$$1 + R_p = \frac{\sum_{i=1}^{n} F_{i1} Z_i - D(n)}{W} = \sum_{i=1}^{n} \left[\frac{F_{i1}}{F_{i0}}\right]\left[\frac{F_{i0} Z_i}{W}\right] - \frac{D(n)}{W} \tag{12}$$

Rewriting,

$$1 + R_p = \sum_{i=1}^{n} (1 + R_i) q_i - C(n) = 1 + \sum_{i=1}^{n} R_i q_i - C(n) \tag{13}$$

where

$1 + R_i = F_{i1}/F_{i0}$ = one plus rate of return on asset i
$q_i = F_{i0} Z_i / W$ = proportion invested in asset i
$C(n) = D(n)/W$ = transactions costs relative to wealth
W = wealth

Thus, the rate of return on the portfolio is

$$R_p = \sum_{i=1}^{n} R_i q_i - C(n) \tag{14}$$

$D(n)$ represents total transactions costs for a portfolio composed of n securities. The nature of cost functions (linear, concave, convex) is discussed below. Dividing total transactions costs by wealth gives $C(n)$, transactions costs per unit of wealth. In the case of n assets with mean return of μ_a, standard deviation of σ_a, and correlation coefficient of r, the expected return μ_p and standard deviation σ_p on the portfolio are

$$\mu_p = E[R_p] = \mu_a - C(n) = \mu_a - \tfrac{D(n)}{W} \tag{15}$$

$$\sigma_p = \sigma_a \sqrt{r + \tfrac{1-r}{n}} \tag{16}$$

The expected return on the portfolio is reduced by the transaction costs $C(n)$. However, the portfolio standard deviation will not be affected by the transactions costs.

There are many possible shapes of the total cost function shown in Figure 3. One possibility is a flat cost function with an initial overhead cost and no additional costs. In this case the number of different securities does not affect the transactions costs. A second possibility is a linear cost function, where the incremental cost per security is a constant. Third, with a convex cost function, the incremental cost per security increases because of diseconomies of more securities. Fourth, for a concave cost function, the incremental cost per security is decreasing; there are incremental economies of portfolio size.

Figure 4 shows a cost function which is initially concave and then becomes convex. That is, there are initially incremental economies of scale. Then, incremental diseconomies of scale appear.

The Efficient Frontier of Risky Assets

Figure 5 shows a set of possible portfolio choices of risky assets available to investors with and without transaction costs. Notice that transaction costs reduce the mean returns to investors. As the number of risky securities in the portfolio increases, the portfolio standard deviation goes down. However, the mean return decreases because of higher total transaction costs.

In Figure 5, the optimal portfolio of risky assets with transaction costs depends upon the investor's degree of risk aversion. Investors with a low level of risk aversion might invest in a portfolio of a single risky asset. Highly risk averse investors might hold the minimum standard deviation portfolio N.

The slope of the efficient frontier depends upon the type of cost function. The following Proposition is proved below.

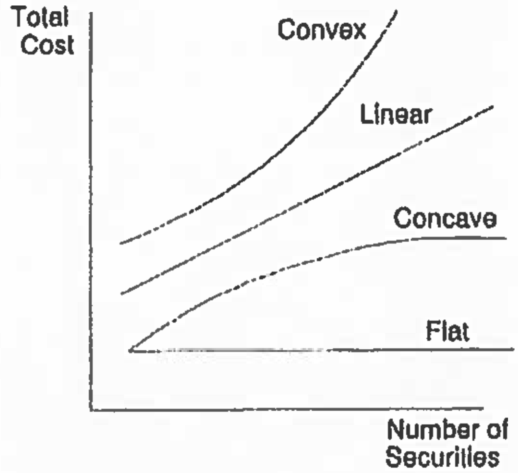

Figure 3: Types of Cost Functions

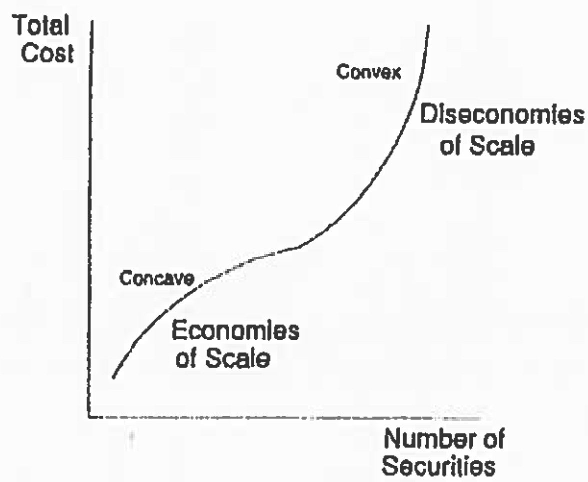

Figure 4: Cost Function with Concave and Convex Sections

PROPOSITION. The efficient frontier has the typical concave shape as shown in Figures 5, 6, or 7 if the total cost function is linear or convex.

Proof. The slope of the efficient frontier is $d\mu_p/d\sigma_p$. We prove that $d\mu_p/d\sigma_p > 0$ and $d^2\mu_p/d(\sigma_p)^2 < 0$ if the total cost function is linear or convex, $dC(n)/dn \geq 0$ and $d^2C(n)/dn^2 \geq 0$.

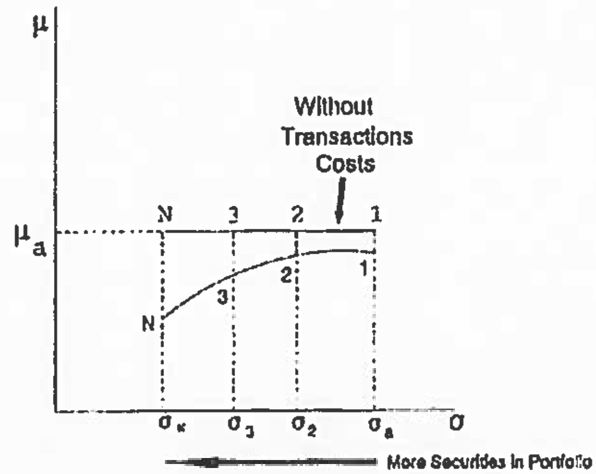

Figure 5: Impact of Transactions Costs

From equation (16), solve for n, the number of average assets.

$$n = \frac{\sigma_a^2(1-r)}{\sigma_p^2 - \sigma_a^2 r} \tag{17}$$

Clearly, $dn/d\sigma_p < 0$ and $d^2n/d(\sigma_p)^2 > 0$.

From equation (15),

$$\frac{d\mu_p}{d\sigma_p} = -\frac{dC(n)}{dn}\frac{dn}{d\sigma_p} \tag{18}$$

Since $dC(n)/dn \geq 0$ for a linear or convex total cost function and $dn/d\sigma_p < 0$ from equation (17), then $d\mu_p/d\sigma_p \geq 0$.

$$\frac{d^2\mu_p}{d(\sigma_p)^2} = -\frac{d^2C(n)}{dn^2}\left[\frac{dn}{d\sigma_p}\right] - \frac{dC(n)}{dn}\left[\frac{d^2n}{d(\sigma_p)^2}\right] \tag{19}$$

If the total cost function is linear or convex, $dC(n)/dn \geq 0$ and $d^2C(n)/dn^2 \geq 0$. From equation (17), $dn/d\sigma_p < 0$ and $d^2n/d(\sigma_p)^2 > 0$. Therefore, $d^2\mu_p/d(\sigma_p)^2 < 0$.

Linear or convex total cost functions are only sufficient conditions for the efficient frontier to be concave. The efficient frontier may still be concave if the total cost function is concave, i.e., $d^2C(n)/dn^2 < 0$.

Wealth Economies

Wealth economies occur if transactions costs per dollar of wealth decrease as investor wealth increases. These economies occur if wealthier investors receive

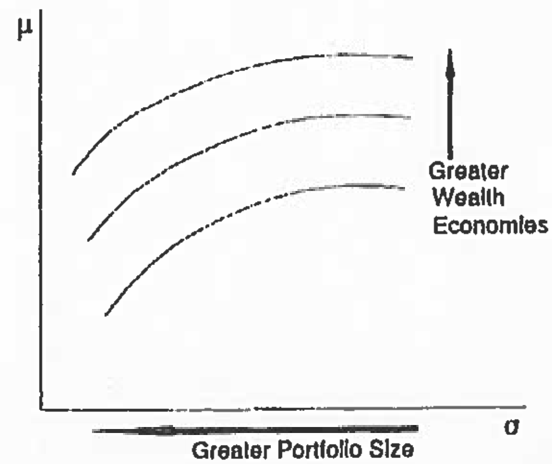

Figure 6: Wealth Economics

more favorable brokerage costs per dollar invested. Also, larger investors may be able to spread information costs over a larger dollar investment.

Figure 6 shows wealth economies. Wealth economies cause an upward shift in the efficient frontier since the ratio $D(n)/W$ is smaller. Larger portfolio size results, other things constant.

Large trades may have an unfavorable impact upon price. Large investors may pay premium prices when buying and may receive low prices when selling. To help reduce the impact of individual trades upon price, the so-called fourth market has developed. In this market, institutions trade anonymously by computer. Buyers and sellers enter their bids or offers into the computer system and interested parties can take the other side of the transaction for all or part of a stated volume of shares.

Adding the Risk-Free Asset

Besides being able to buy n risky assets with transactions costs, investors are assumed in this section to be able to lend and borrow at the risk-free rate, R_f, without transactions costs. The time 0 cash flows are

$$-W = -\sum_{i=1}^{n+1} F_{i0} Z_i \tag{20}$$

The time 1 cash flows are

$$\sum_{i=1}^{n+1} F_{i1} Z_i - D(n) \tag{21}$$

One plus the return on the portfolio is

$$1 + R_p = \frac{\sum_{i=1}^{n+1} F_{i1} Z_i - D(n)}{W} = \sum_{i=1}^{n+1} \left[\frac{F_{i1}}{F_{i0}}\right]\left[\frac{F_{i0} Z_i}{W}\right] - \frac{D(n)}{W} \quad (22)$$

If $1 + R_i = F_{i1}/F_{i0}$ and $w_i = F_{i0} Z_i / W$, then

$$1 + R_p = \sum_{i=1}^{n} R_i w_i + w_{n+1} R_f - \frac{D(n)}{W} \quad (23)$$

Let

α = proportion of wealth invested in risky assets.

$(1 - \alpha)$ = proportion of wealth invested in the risk-free asset.

Then

$$R_p = \alpha \sum_{i=1}^{n} R_i v_i + (1 - \alpha) R_f - \frac{D(n)}{W} \quad (24)$$

where

$$\sum_{i=1}^{n} v_i = 1 \quad (25)$$

The expected return on the portfolio is

$$E(R_p) = \mu_p = \alpha E\left[\sum_{i=1}^{n} R_i v_i\right] + (1 - \alpha) R_f - \frac{D(n)}{W} \quad (26)$$

Since

$$E \sum_{i=1}^{n} R_i v_i = \mu_a \quad (27)$$

the portfolio mean is

$$\mu_p = \alpha \mu_a + (1 - \alpha) R_f - \frac{D(n)}{W} \quad (28)$$

The portfolio standard deviation is

$$\alpha \sigma_n = \alpha \sigma_a \sqrt{r + \frac{1 - r}{n}}. \quad (29)$$

The shape of the efficient frontier depends upon $D(n)/W$, in particular $d[D(n)/W]/d\alpha$. The following discussion considers cases where cost is proportional to α, independent of α, and cases of partial dependence.

α, Cost Functions, and the Shape of the Efficient Frontier

The shape of the efficient frontier depends upon the cost function and α. We assume that the risk-free asset can be acquired without transactions costs. If all wealth is invested in risky assets ($\alpha = 1$), total transactions costs are $D_1(n)$. If less than 100% of the portfolio is invested in risky assets, transactions costs are typically reduced. There are several possibilities. In one case, the transactions costs are simply reduced proportionally to $\alpha D_1(n)$. In another case, transactions costs are not affected at all. They are independent of α. In general, the transactions may be some nonproportional function of the proportion invested in the risky assets. Each of these three cases is examined in the discussion to follow.

Several interesting results are shown. First, if there are wealth economies (i.e., if $D(n)/W$ decreases as W increases), wealthier investors hold larger portfolios. Consequently, wealth economies by themselves cause portfolio separation to break down. Second, with proportional costs, investors with a given wealth hold the same portfolio; portfolio separation holds for particular wealth levels. Third, for costs independent of α, portfolio separation does not hold for a particular wealth level. For partial dependence of costs upon α, portfolio separation does not hold.

Cost is Proportional to α

Denote cost with a portfolio fully invested in n risky assets (i.e., $\alpha = 1$) as $D_1(n)$. α proportion of wealth is invested in risky assets at a cost of $\alpha D_1(n)$, and $1 - \alpha$ in risk-free assets with no cost. Total transactions costs are proportional to α, i.e., $\alpha D_1(n)$. Then,

$$\begin{aligned} \mu_p &= \alpha\mu_a + (1-\alpha)R_f - \frac{\alpha D_1(n)}{W} \\ &= \alpha[\mu_a - D_1(n)/W] + (1-\alpha)R_f \end{aligned} \tag{30}$$

Algebraically, an investor would like to maximize the slope, m, of the rays through the risk-free rate and the various possible portfolios. This slope represents the market price of risk.

$$m = \frac{1}{\sigma_a}\left[\frac{\mu_a - R_f - C(n)}{\sqrt{r + \frac{1-r}{n}}}\right] \tag{31}$$

This case is shown in Figure 7 for investors with two different wealth levels W_A and W_B, $W_A < W_B$. For W_A, the efficient frontier of risky assets is $\mu_a - D_{1,A}(n)/W_A$; for W_B, it is $\mu_a D_{1,B}(n)/W_B$. The figure assumes wealth economies so that $D_{1,A}(n)/W_A > D_{1,B}(n)/W_B$ and a resulting higher efficient frontier of risky assets for W_B.

For each wealth level, investors pick portfolios on the highest ray going through the risk-free asset and tangent to the efficient frontier for their wealth level. If the efficient frontier of risky assets is concave, there is a unique portfolio of assets

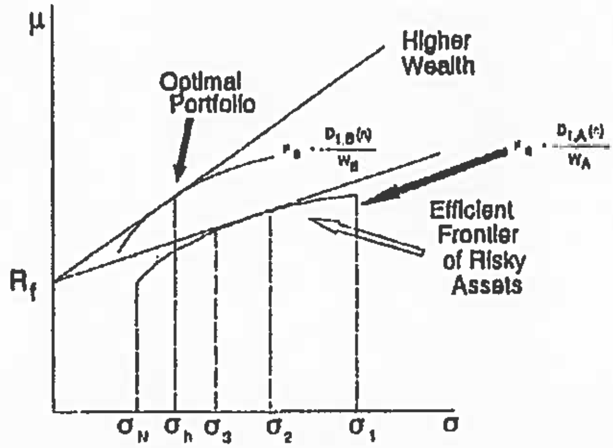

Figure 7: The Optimal Portfolio with Transactions Costs Proportional to Alpha

for that wealth level. But wealth economies cause wealthier investors to hold a larger portfolio than poorer investors. Thus, wealth economies per se cause portfolio separation to break down. In this world of wealth-dependent portfolios, equilibrium is described by the Generalized CAPM of Levy (1978).

With transaction costs proportional to α, the optimal portfolio for a particular level of wealth will generally include only some of the N risky securities available to investors. There is an optimal portfolio size which balances the incremental gains from diversification against the incremental costs of adding securities. At the optimum, the marginal benefit from diversification equals the marginal cost of adding a security.

For a particular wealth level, it can be shown that an increase in the excess return, $\mu_a - R_f$ or a reduction in the correlation coefficient, r, increases portfolio size, ceteris paribus. Portfolio size decreases if the borrowing rate increases or the mean return decreases. In addition, changes over time in portfolio standard deviation should not alter optimal portfolio size. If the overall market becomes riskier, portfolio size does not change.

Figure 7 clearly shows that the optimal portfolio size depends upon investor wealth if there are wealth economies (i.e., if $D_{1,A}(n)/W_A > D_{1,B}(n)/W_B$ for $W_A < W_B$). Wealthier investors hold larger portfolios if there are wealth economies.

For transactions costs proportional to α, portfolio separation holds for a particular wealth level; all investors with the same wealth hold the same portfolio of risky assets, independent of investor risk preferences. But with wealth economies, wealthier investors typically hold larger portfolios than smaller investors. Thus, global separation does not occur with proportional transactions costs if economies of wealth occur.

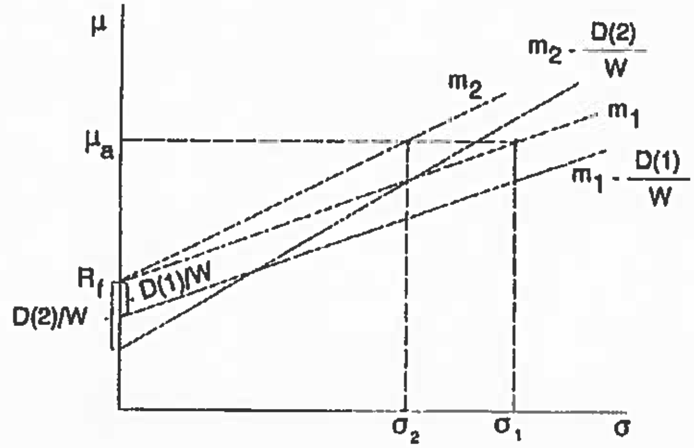

Figure 8: Cost Independent of Alpha

Cost Is Independent of α

Without transactions costs, Figure 2 showed that investors hold as many assets as possible. For a given wealth level, Figure 8 shows the case when transactions costs are a function of n but independent of α. The highest efficient rays without transactions costs are $m_1, m_2, \ldots$. Since, transactions costs are fixed as the proportion, α, invested in risky assets varies, transactions costs cause a parallel shift of $D(n)/W$ in the efficient rays, with a larger shift as n increases.

If borrowing is allowed, the efficient frontier is a continuous, convex set of line segments as shown in Figure 9. This occurs because the vertical intercept of successive efficient rays is lower (higher cost) and the slope higher (because σ_n is smaller) as n increases. If borrowing is not allowed, the efficient frontier is a concave set of discontinuous line segments. In either case, the efficient frontier is not a straight line. Therefore, the choice of optimal portfolio will depend upon investor preferences and separation does not hold even for a particular wealth level. Greater investor risk aversion implies more securities in the portfolio.

Thus, portfolio separation does not hold with transactions costs independent of α. Investor portfolios will depend upon individual risk preferences.

Cost is a Nonproportional Function of α

Brokerage costs may quite plausibly be a proportional function of α. Information costs are more likely to be independent of α. Since investors typically incur both brokerage and information costs, total costs are probably a nonproportional function of α.

Instead of parallel shifts in efficient rays (as in the case of costs independent of α), nonproportional costs cause nonparallel shifts, creating a nonlinear efficient frontier and portfolios which may depend upon investor preferences for a particular

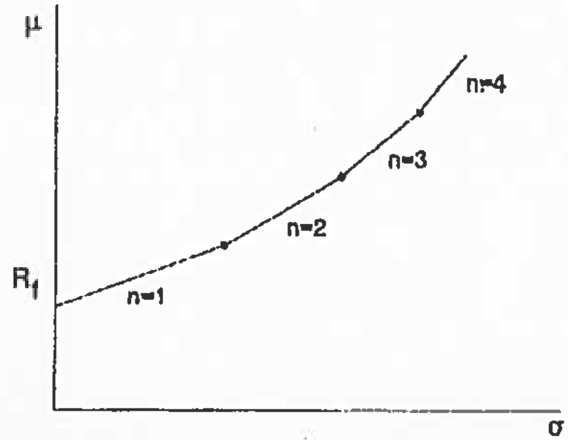

Figure 9: Efficient Frontier with Costs Independent of alpha

wealth level. Thus, portfolio separation may not hold for individual wealth levels. The concavity or convexity of the efficient frontier (and portfolio separation) depend upon the parameters of the cost function.

IV. AVERAGE SECURITIES AND TRANSACTION COSTS: BROKERAGE COSTS

Transactions costs have two primary components: information costs and brokerage costs. Objective evidence on information costs is not available. Considerable evidence about brokerage costs is available and is used below to examine the impact of brokerage costs upon portfolio size.

Discount Brokerage Cost Schedules

Investors have a variety of brokers from which to choose, including discount brokers and full service brokers. Pure discount brokers charge for execution only and charge the lowest fees. Full service brokers charge higher fees reflecting execution costs as well as a charge for research and advice. We contacted a number of full service brokers. All were unwilling or unable to provide brokerage commission schedules. Theoretically, these brokers do not have a set schedule of fees, since the commissions are supposed to be negotiable.

Discount brokers provided us with brokerage commission schedules. Typical commissions are shown in Table 2. There were two types of brokerage schedules—large volume brokers and small volume brokers. For large volume brokers, brokerage costs are two cents per share. As shown below this type of commission results in a parallel downward shift in the efficient frontier. Large investors will hold as many securities as possible and portfolio separation applies. For small

Table 2: Discount Brokers. Brokerage Fees as a Proportion of Amount Invested

	SIZE OF TRANSACTION			
Brokers which charge a flat fee plus % of principal:	$100,000	$500,000	$1 M	$5 M
Nations Bank	.0055	.0051	.00505	.00501
New England Securities	.0095	.0076	.00755	.00751
Lehigh Securities	.00312	.00129	.00115	.00103
Quick and Reilly	.00205	.00109	.000885	.000721
Charles Schwab	.00265	.00141	.00116	.000951
Bull and Bear	.00212	.00113	.000924	.000761
Fidelity	.00265	.00141	.00116	.000951
First Union	.00240	.00128	.00104	.000848
Brokers which charge on a per-share basis:				
Jack White (2 cents/share)	.044	.044	.044	.044
Lombard (2 cents/share)	.044	.044	.044	.044
AccuTrade (3 cents/share)	.067	.067	.067	.067

The final three brokers in the table charge on a per share basis. Percentage principal costs are calculated assuming an average share price of $45.

volume brokers, brokerage costs are a function of dollars invested. As the dollar amount invested increases, brokerage costs as a percent tend to decrease. This type of cost function typically results in a concave efficient frontier. For a particular wealth level, portfolio separation holds. But investors with different wealth levels hold different portfolios.

Large Volume Discount Brokers

The large volume discount brokers require a minimum number of shares per transaction, typically 1,500 shares. For these large volume brokers, the costs are X cents per share, typically 2 or 3 cents per share. With this type of cost function, the price per share significantly impacts total brokerage costs. However, if we make the strong assumption of constant price per share as the number of securities in the portfolio increases, the total cost is simply X times the number of shares, which equals X(wealth)/(price per share). The number of different securities in the portfolio does not affect total cost. The total cost function, $D(n)$, is flat. As wealth increases, the number of shares increases and total cost increases.

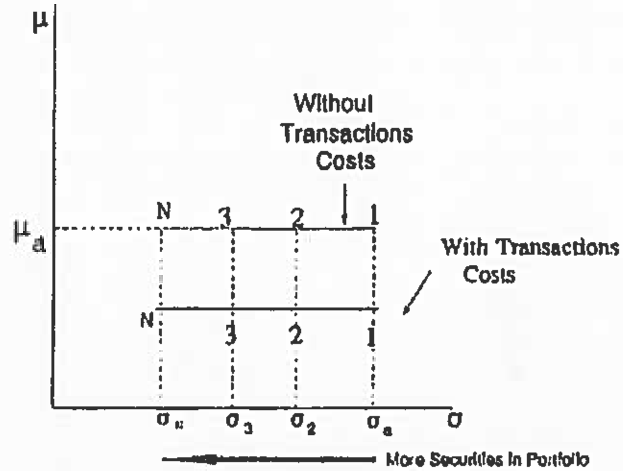

Figure 10: Impact of Transactions costs for Large Volume Discount Brokers

Brokerage cost per dollar of wealth, $C(n)$, is

$$C(n) = \frac{X[S]}{W} = \frac{X[W/P]}{W} = \frac{X}{P} \tag{32}$$

where X = brokerage cost per share; S = the number of shares; P = price per share; W = an investor's total wealth.

For large volume discount brokers, brokerage cost per dollar of wealth, $C(n)$, is a function of the cost per share, X, and the price per share, P. Brokerage cost per dollar of wealth is independent of the level of wealth or the number of securities in the portfolio. If the price per share is $45 and the commission is 2 cents per share, the cost per dollar of wealth is approximately $.00044, about 4.4 basis points.

For large volume discount brokers, brokerage commissions shift the efficient frontier downward by the commission percent assuming a constant price per share.[9] This type of parallel shift in the efficient frontier is shown in Figure 10 and is covered in more detail in our discussion of mutual funds, which most likely deal with these large volume discount brokers. Investors with this type of cost function are fully diversified. They hold the entire universe of risky securities.

Small Volume Discount Brokers

The brokerage commissions charged by small volume discount brokers vary. Some brokers, such as Nations Bank, have a simple linear brokerage cost function. The

[9] As the share price decreases, the percent commission increases and the efficient frontier shifts downward.

brokerage cost for each stock purchased is \$50 plus .5% of the dollar value of the transaction. Thus, if three different stocks were purchased, the total cost would be \$150 plus .5% of the total dollar value of the transactions.

Other brokers have a more complicated schedule. The total commission is a fee for each stock purchased, $a(j)$, plus a percent, $b(j)$, of the total dollar value of the transaction.[10] The parameter j represents the dollar size of the transaction. The function is piecewise linear with the parameters $a(j)$ and $b(j)$ fixed for particular intervals. As the size of the transaction increases the fixed fee increases, but the percent fee declines. Examples of this type of commission schedule are shown in Table 3 for Quick & Reilly, New England Securities, and Charles Schwab. We designate the various fixed dollar fees as $a(j)$ and the percent cost as $b(j)$.

An example of the cost function for Quick & Reilly is shown in Table 4. If the dollars invested in a particular security are between 0 and 2,500 dollars, $a(j)$ is \$22 and $b(j)$ is 0.014. The cost is \$22 + (0.014)(\$). If the dollars invested in a particular security are between 2,501 and 6,000 dollars, $a(j)$ is \$38 and $b(j)$ is 0.0045. The cost is \$38 + (0.0045)(\$), etc. The cost for investing in one stock is $D(1)$, where

$$D(1) = a(j) + b(j)(\$) \tag{33}$$

The percentage cost function for Quick & Reilly is shown in Table 5. The percentage cost function is the total cost divided by the dollars invested. The percentage cost for investing in one stock is $C(1)$, where

$$C(1) = \frac{a(j)}{\$} + b(j) \tag{34}$$

Total Cost for a Portfolio of *n* Securities

Suppose an investor invests his wealth W in equal proportions in each of n securities. j dollars are invested in each security. (i.e., $j = W/n$). The cost of investing in a particular security has two components. The charge per stock is $a(j)$. The level of $a(j)$ depends upon the dollar amount invested in each stock. As the dollar size of the transaction increases, $a(j)$ increases. The other component, $b(j)$, is a percent of the dollars invested in that security. $b(j)$ decreases in steps as the amount invested in each stock increases. The cost of investing in each of the n securities is $a(j) + b(j)(W/n)$.

The total cost $D(n)$ of investing in n securities is the sum of all the individual n security costs. Since equal dollar amounts are invested in each security, the total cost for n securities is n times the cost for an individual security.

$$D(n) = (n)a(j) + b(j)W \tag{35}$$

[10]Since investors are assumed to have total wealth of W, invested equally in n securities, the investment in each security is W/n. The brokerage cost interval depends upon W/n.

Table 3: Brokerage Schedules

QUICK & REILLY

low-end	high-end	flat fee $a(j)$	percent fee $b(j)$
0	2,500	22	.01400
2,501	6,000	38	.00450
6,001	22,000	59	.00250
22,001	50,000	77	.00170
50,001	500,000	120	.00085
500,001		205	.00068

NEW ENGLAND SECURITIES

low-end	high-end	flat fee $a(j)$	percent fee $b(w/n)$
0	2,500	50	.01400
2,501	10,000	50	.01300
10,001	25,000	50	.01200
25,001	50,000	50	.01100
50,001	100,000	50	.00900
100,001		50	.00750

CHARLES SCHWAB

low-end	high-end	flat fee $a(j)$	percent fee $b(j)$
0	2,500	30	.01700
2,501	6,250	56	.00660
6,251	20,000	76	.00340
20,001	50,000	100	.00220
50,001	500,000	155	.00110
500,001		255	.00090

The cost per dollar of wealth is $C(n)$.

$$C(n) = \frac{D(n)}{W} = \frac{(n)a(j)}{W} + b(j) \tag{36}$$

The impact of portfolio size on transactions costs depends upon the parameters $a(j)$, $b(j)$, and total wealth W. Figures 11, 12, and 13 show total and percent transactions costs for three discount broker as the number of securities in the

Table 4: Total Cost Functions for Quick & Reilly

QUICK & REILLY		
low-end	high-end	TOTAL COST FUNCTION $a(j) + b(j)(\$)$
0	2,500	22 + .01400($)
2,501	6,000	38 + .00450($)
6,001	22,000	59 + .00250($)
22,001	50,000	77 + .00170($)
50,001	500,000	120 + .00085($)
500,001		205 + .00068($)

Table 5: Percent Cost Functions for Quick & Reilly

QUICK & REILLY		
low-end	high-end	PERCENT COST FUNCTION $a(j)/(\$) + b(j)$
0	2,500	22/($) + .01400
2,501	6,000	38/($) + .00450
6,001	22,000	59/($) + .00250
22,001	50,000	77/($) + .00170
50,001	500,000	120/($) + .00085
500,001		205/($) + .00068

portfolio increases for several wealth levels. Typically, the total transactions costs increase and percentage costs decrease as portfolio size increases.

Figures 14, 15, and 16 show total and percent transactions costs as the number of securities and total wealth change. As the number of securities increases, total costs and percent costs increase. As wealth increases, total costs increase, but percent costs are reduced.

Figures 17, 18, and 19 show efficient frontiers for several wealth levels for three small volume discount brokers. The efficient frontiers are usually concave. The exceptions occur for lower wealth levels around breakpoints in the cost function. At these breakpoints, the intercept and the slope of the cost function change.

Approximating the Cost Functions of Small Volume Discount Brokers

The earlier evidence showed that small volume discount brokerage cost functions are of two types. (1) A simple linear function with a constant intercept and slope. (2) A series of linear segments (where intercept and slope can differ for each

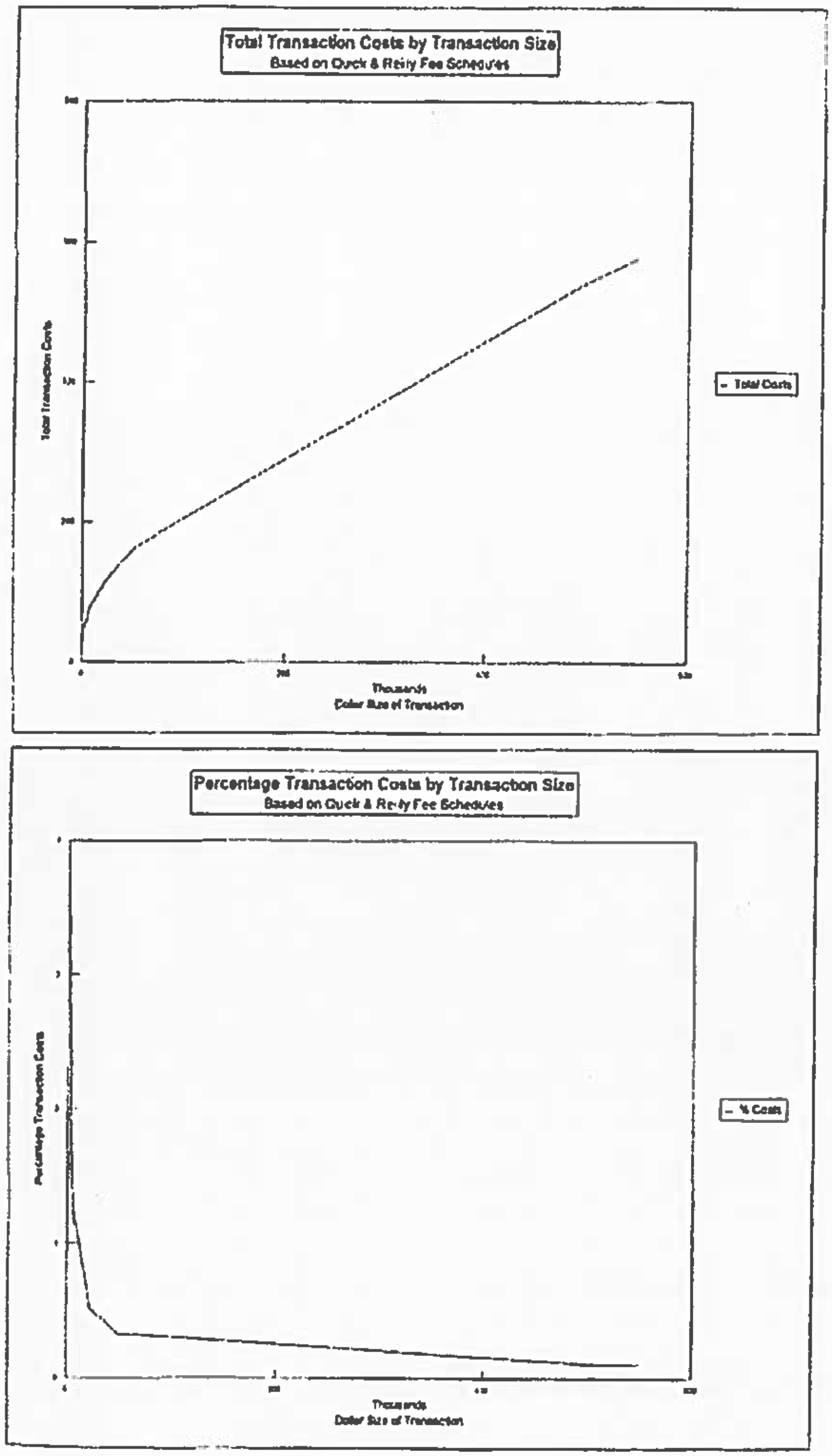
Total Transaction Costs by Transaction Size
Based on Quick & Reily Fee Schedules
Total Costs
Percentage Transaction Costs by Transaction Size
Based on Quick & Reily Fee Schedules
% Costs

Figure 11

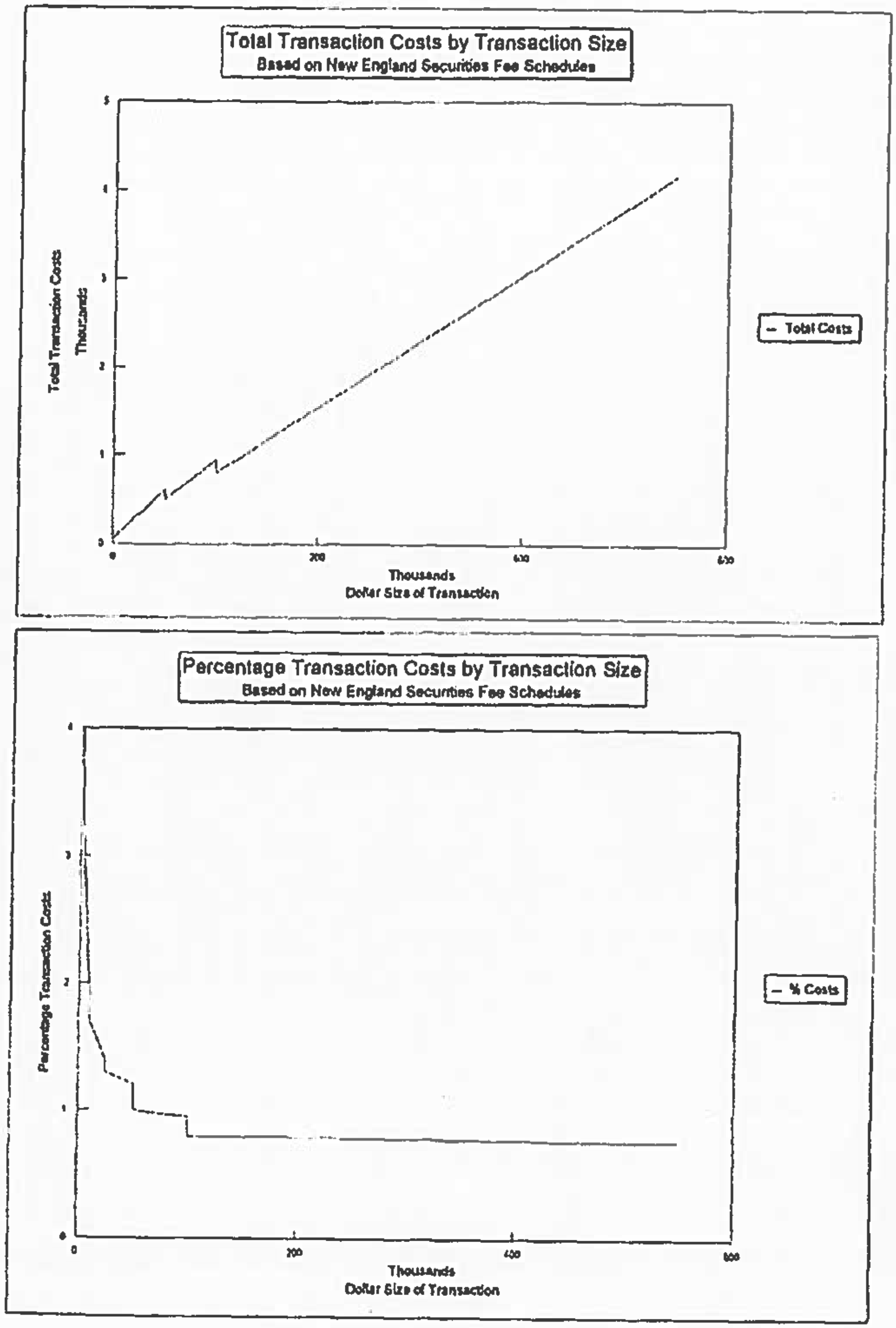
Total Transaction Costs by Transaction Size
Based on New England Securities Fee Schedules
Total Transaction Costs
Thousands
0
1
2
3
4
5
0
200
400
600
Thousands
Dollar Size of Transaction
Total Costs
Percentage Transaction Costs by Transaction Size
Based on New England Securities Fee Schedules
Percentage Transaction Costs
0
1
2
3
4
0
200
400
600
Thousands
Dollar Size of Transaction
% Costs

Figure 12

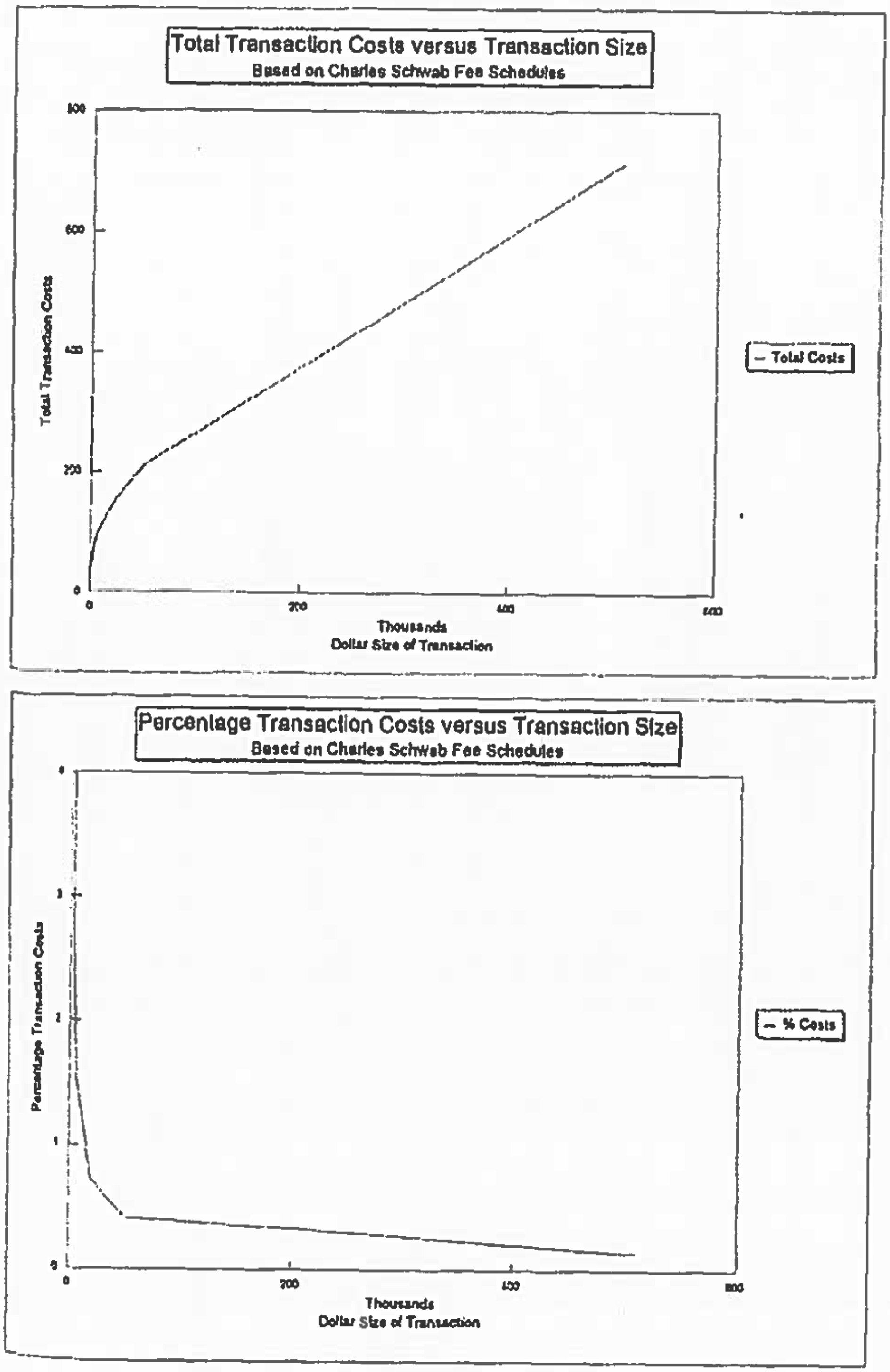
Total Transaction Costs versus Transaction Size
Based on Charles Schwab Fee Schedules
Total Transaction Costs
Thousands
Dollar Size of Transaction
Total Costs
Percentage Transaction Costs versus Transaction Size
Based on Charles Schwab Fee Schedules
Percentage Transaction Costs
Thousands
Dollar Size of Transaction
% Costs

Figure 13

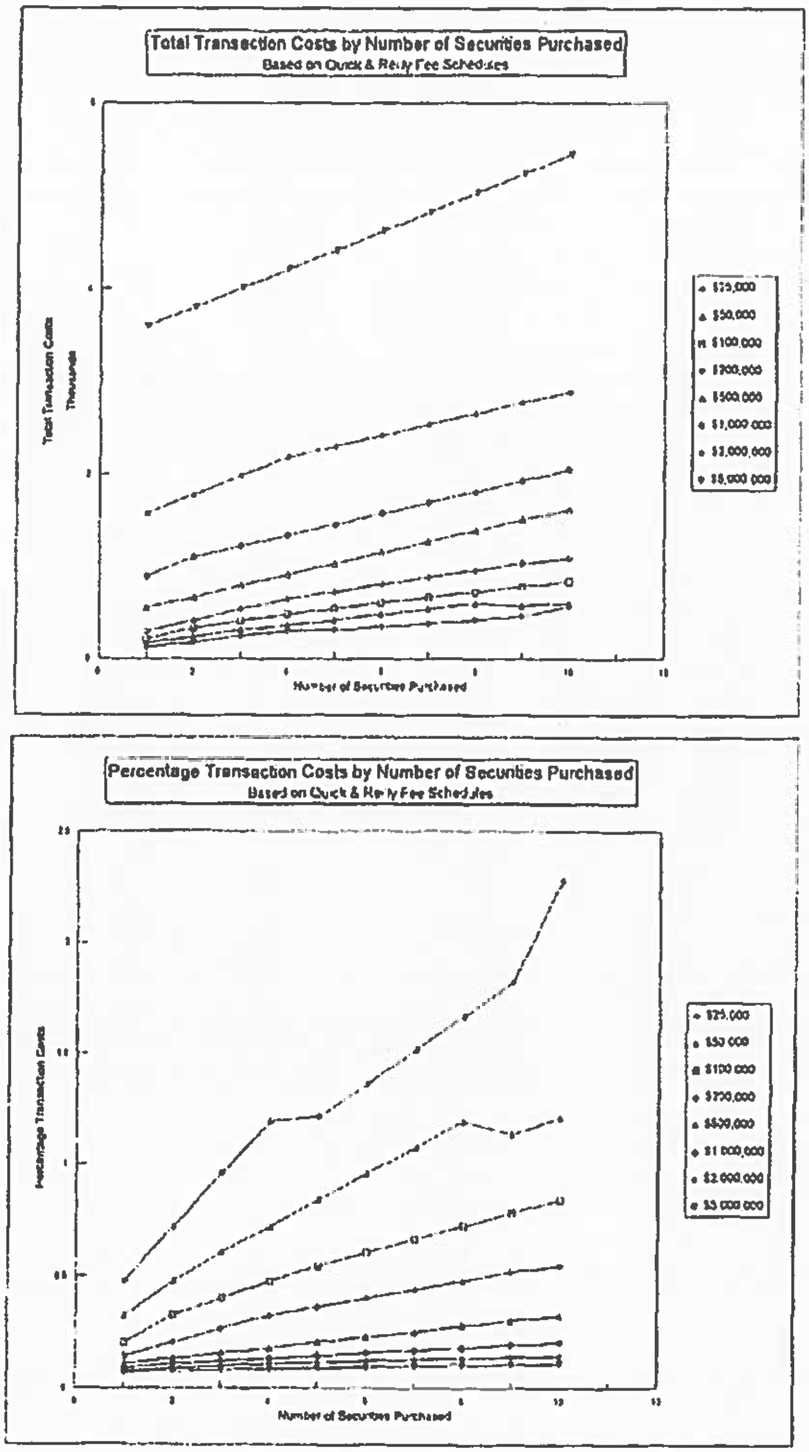
Total Transaction Costs by Number of Securities Purchased
Based on Quick & Reilly Fee Schedules
Total Transaction Costs
Thousands
Number of Securities Purchased
$25,000
$50,000
$100,000
$200,000
$500,000
$1,000,000
$2,000,000
$5,000,000
Percentage Transaction Costs by Number of Securities Purchased
Based on Quick & Reilly Fee Schedules
Percentage Transaction Costs
Number of Securities Purchased
$25,000
$50,000
$100,000
$200,000
$500,000
$1,000,000
$2,000,000
$5,000,000

Figure 14

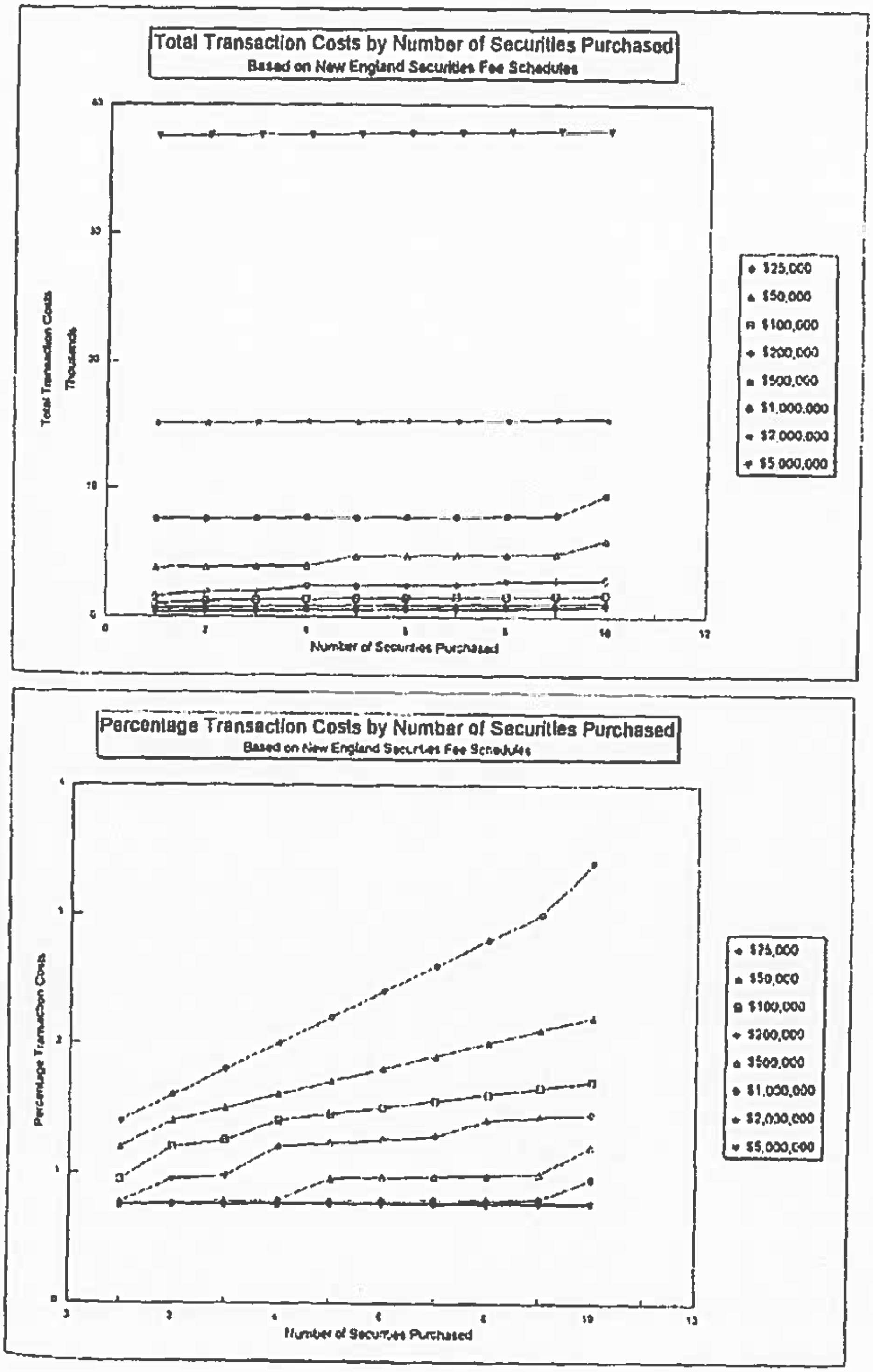
Total Transaction Costs by Number of Securities Purchased
Based on New England Securities Fee Schedules
Total Transaction Costs
Thousands
Number of Securities Purchased
$25,000
$50,000
$100,000
$200,000
$500,000
$1,000,000
$2,000,000
$5,000,000
Percentage Transaction Costs by Number of Securities Purchased
Based on New England Securities Fee Schedules
Percentage Transaction Costs
Number of Securities Purchased
$25,000
$50,000
$100,000
$200,000
$500,000
$1,000,000
$2,000,000
$5,000,000

Figure 15

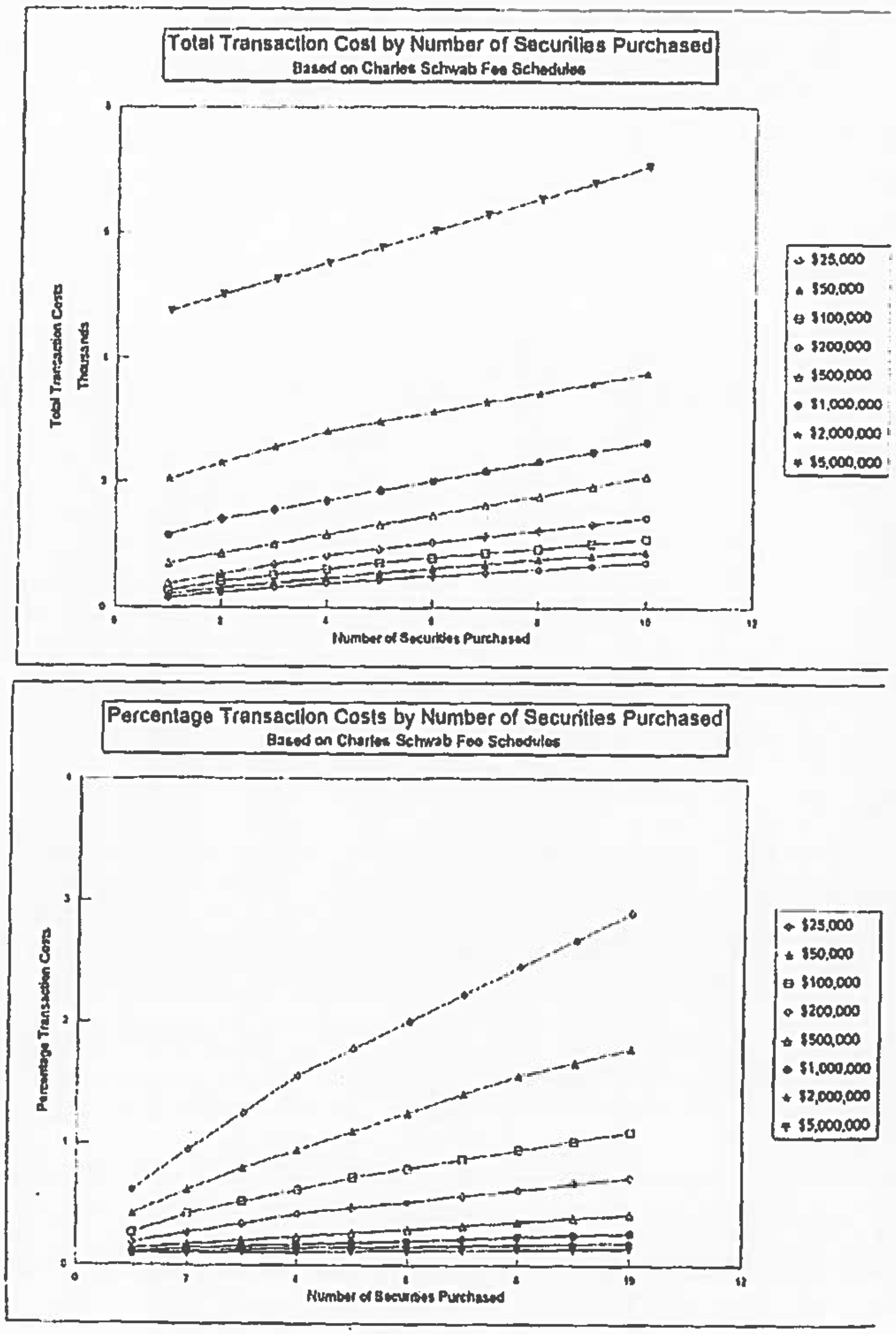
Total Transaction Cost by Number of Securities Purchased
Based on Charles Schwab Fee Schedules
Total Transaction Costs
Thousands
Number of Securities Purchased
$25,000
$50,000
$100,000
$200,000
$500,000
$1,000,000
$2,000,000
$5,000,000
Percentage Transaction Costs by Number of Securities Purchased
Based on Charles Schwab Fee Schedules
Percentage Transaction Costs
Number of Securities Purchased
$25,000
$50,000
$100,000
$200,000
$500,000
$1,000,000
$2,000,000
$5,000,000

Figure 16

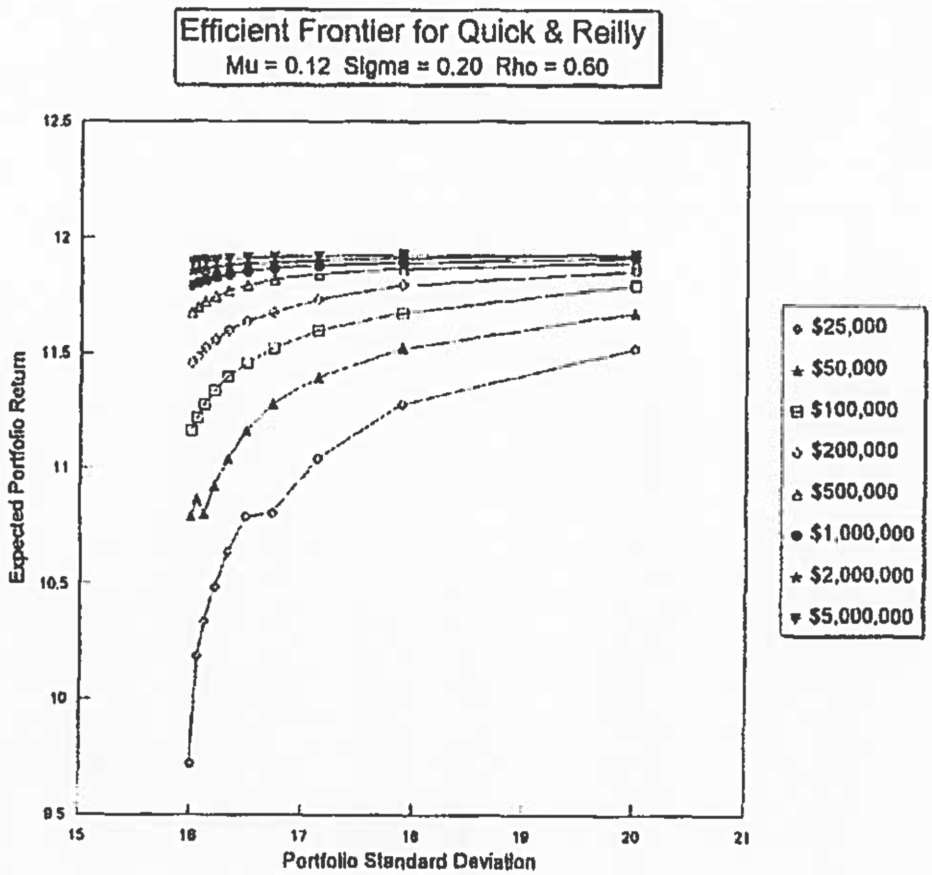

Figure 17

segment). With a simple linear function, optimal portfolio size can be derived as shown below. With a series of linear segments, the question of optimal portfolio requires numerical solutions. However, this case can be closely approximated by a simple linear function with fixed cost a and percent cost b.

Suppose that fixed cost, $a(j)$ and percent cost, $b(j)$, are independent of the amount invested, W/n. The fixed cost is simply a and the percent cost is simply b. Then the total portfolio transaction cost for investing W/n in each of n securities is $D(n)$, where

$$D(n) = na + bW \tag{37}$$

The portfolio transaction cost as a percent of the amount invested is $C(n)$, where

$$C(n) = \frac{na}{W} + b \tag{38}$$

As proved earlier, when the cost function is a linear function of the number of securities in the portfolio, the efficient frontier of risky assets is concave for a

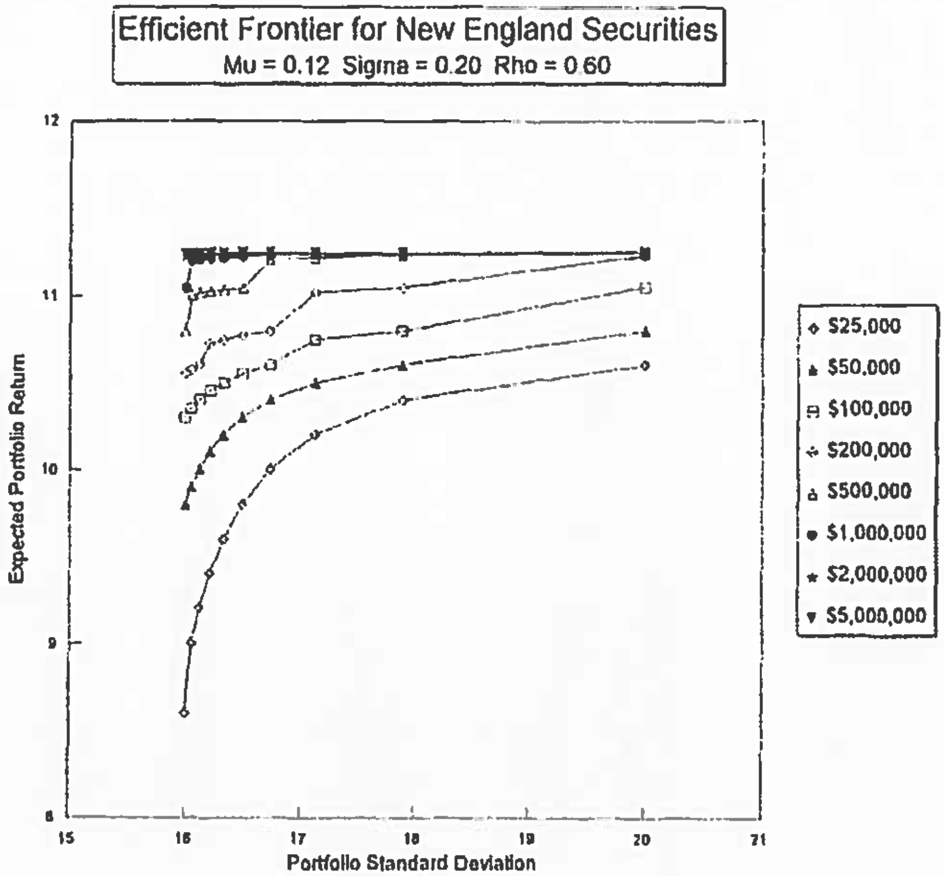

Figure 18

given wealth level and for each wealth level portfolio separation holds. Thus, if the fixed cost and percent cost parameters are independent of wealth, the efficient frontier of risky assets is concave and portfolio separation holds for each wealth level.

Linear Cost Functions and Optimal Portfolio Size

Assume a linear cost function, $C(n) = na/W + b$, with transactions costs proportional to α. The optimal-sized portfolio of risky assets with transactions costs proportional to α maximizes m in the following equation.

$$m = \frac{\mu_a - R_f - \left[\frac{an}{W} + b\right]}{\sigma_a\sqrt{r + \frac{1-r}{n}}} \qquad (39)$$

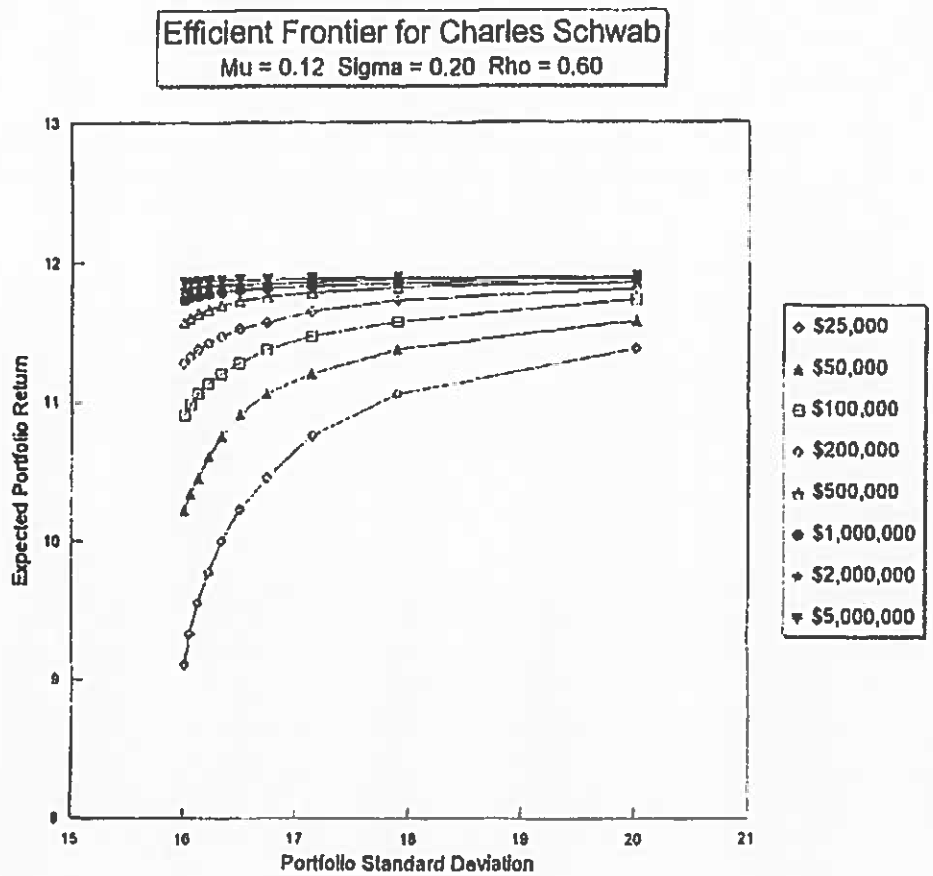

Figure 19

Take the derivative with respect to n, set equal to zero, and solve by the quadratic formula for the optimal portfolio size n. See Appendix B.

$$n = \frac{-3 + \sqrt{9 + \left[\frac{8r}{1-r}\right]\left[\frac{\mu_a - R_f - b}{a/W}\right]}}{\frac{4r}{1-r}} = \frac{-3 + \sqrt{9 + \left[\frac{8r}{1-r}\right][KW]}}{\frac{4r}{1-r}} \tag{40}$$

where $K = (\mu_a - R_f - b)/a$. K will be used later on.

From this equation, optimal portfolio size is inversely related to (1) percent cost, b, $(dn/db < 0)$, (2) fixed cost, a, $(dn/da < 0)$, and (3) the correlation coefficient, r $(dn/dr < 0)$. Optimal portfolio size is positively related to (1) wealth, W and (2) mean return minus the risk-free rate, $\mu_a - R_f$, $(dn/d[\mu_a - R_f] > 0)$.

Upper Bounds on Portfolio Size with Linear Cost Function

Upper bounds on the value of optimal portfolio size can be found for several cases. Since portfolio size n and the correlation coefficient r are inversely related, a low value of the correlation coefficient provides an upper bound upon portfolio size compared to larger values of the correlation coefficient. See the Appendix for proof that $dn/dr < 0$.

The following tables show optimal portfolio size for a linear cost function for correlation coefficients of 0.0, 0.50, and 0.75. The two varying parameters in each table are wealth, W, and K, defined as $(\mu_a - R_f - b)/a$. In each table wealth varies from \$10,000 to \$500,000. The parameter K varies from .0001 to .01. If $\mu_a - R_f - b$ can plausibly be assumed to be .10 or less and a can be assumed to be \$50 or more, then the plausible values of K are .0020 or less.

Correlation Coefficient = 0.0

In the special case where the correlation coefficient between securities is zero (i.e., $r = 0$), the optimal portfolio size n is

$$\begin{aligned} n &= \frac{[\mu_a - R_f - b]\,W}{3a} \\ &= \frac{KW}{3} \end{aligned} \tag{41}$$

Table 6 shows optimal portfolio size for a correlation coefficient of 0.0. If the correlation coefficient exceeds zero, the portfolio size is smaller than this upper bound. In Table 6, some of the portfolio sizes are sizable for larger values of K and W.

Correlation Coefficient = 0.50

When the correlation coefficient is 0.50, the optimal portfolio size is shown in the following equation.

$$\begin{aligned} n &= \frac{-3 + \sqrt{9 + 8\left[\frac{\mu_a - R_f - b}{a/W}\right]}}{4} \\ &= \frac{-3 + \sqrt{9 + 8KW}}{4} \end{aligned} \tag{42}$$

The optimal portfolio size for a correlation coefficient of 0.50 is shown in Table 7. When the correlation coefficient is greater than 0.50, the optimal portfolio size is lower than the number in this upper bound since portfolio size and correlation

Table 6: Optimal Portfolio Size
(Linear Cost Function and Correlation Coefficient = 0.0)

	WEALTH (Thousands of Dollars)					
K	10	20	50	100	200	500
.0001	.3	.7	1.7	3.3	6.7	16.7
.0002	.7	1.3	3.3	6.7	13.3	33.3
.0006	2.0	4.0	10.0	20.0	40.0	100.0
.0008	2.7	5.3	13.3	26.7	53.3	133.3
.001	3.3	6.7	16.7	33.3	66.7	166.6
.0015	5.0	10.0	25.0	50.0	100.0	250.0
.002	6.7	13.3	33.3	66.7	133.3	333.3
.005	16.7	33.3	83.3	166.7	333.3	833.0
.01	33.3	66.7	166.7	333.3	666.7	1,667.

Table 7: Optimal Portfolio Size
(Linear Cost Function and Correlation Coefficient = 0.50)

	WEALTH (Thousands of Dollars)					
K	10	20	50	100	200	500
.0001	.3	.5	1.0	1.6	2.5	4.3
.0002	.5	.9	1.6	2.5	3.8	6.4
.0006	1.1	1.8	3.2	4.8	7.0	11.5
.0008	1.4	2.2	3.8	5.6	8.2	13.4
.001	1.6	2.5	4.3	6.4	9.3	15.1
.0015	2.1	3.2	5.4	7.9	11.5	18.6
.002	2.5	3.8	6.4	9.3	13.4	21.6
.005	4.3	6.4	10.5	15.1	21.6	34.6
.01	6.4	9.3	15.1	21.6	30.9	49.3

coefficient are inversely related. Compare the portfolio sizes for correlations of .50 (in Table 7) with the correlations of 0.0 (in Table 6). Optimal portfolio size is much smaller when the correlation is .50. The difference is much more substantial when *K* and *W* are large, i.e., when brokerage costs are high and wealth is larger.

Table 8: Optimal Portfolio Size
(Linear Cost Function and Correlation Coefficient = 0.75)

	WEALTH (Thousands of Dollars)					
K	10	20	50	100	200	500
.0001	.2	.3	.5	.8	1.2	2.0
.0002	.3	.5	.8	1.2	1.8	2.9
.0006	.6	.9	1.5	2.2	3.2	5.1
.0008	.7	1.1	1.8	2.6	3.7	5.9
.001	.8	1.2	2.0	2.9	4.1	6.7
.0015	1.0	1.5	2.5	3.6	5.1	8.2
.002	1.2	1.6	2.9	4.1	5.9	9.5
.005	2.0	2.9	4.7	6.7	9.5	15.1
.01	2.9	4.1	6.7	9.5	13.5	21.5

Correlation Coefficient = .75

Optimal portfolio size is

$$n = \frac{-3 + \sqrt{9 + [24]\left[\frac{\mu_a - R_f - b}{a/W}\right]}}{16} = \frac{-3 + \sqrt{9 + 24KW}}{16} \qquad (43)$$

Table 8 shows optimal portfolio size for a correlation coefficient of 0.75. Higher correlation coefficients result in lower optimal portfolio sizes.

The tables show that the optimal portfolio size is very sensitive to the correlation coefficient. If the correlation coefficient is zero, the portfolio sizes are large for higher brokerage costs and larger wealth. For a correlation coefficient of .50, the optimal portfolio size shrinks dramatically, especially for higher brokerage costs and larger wealth. For a correlation coefficient of .75, the optimal portfolio size shrinks even more. There is considerable evidence that the average correlation between stocks is 0.50 or larger, implying small optimal portfolio size, especially for smaller investors.[11]

V. MUTUAL FUNDS VERSUS DIRECT INVESTMENT

The following analysis shows the diversification advantage of mutual funds in a world of average assets and transactions costs. A mutual fund allows small investors to achieve diversification benefits at lower costs than direct investment.

[11] See King (1966), Livingston (1977), Meyers (1973).

Brokerage fees for mutual funds are deducted from the net asset value of the fund. In effect, brokerage costs are passed on to mutual fund investors. In addition, funds charge a management fee, usually a percent of assets. The fund management pays its overhead expenses from the management fee.[12]

We will assume that all funds are fully invested in risky assets and pay $k\%$ of the purchase price as brokerage fees. The assumption of brokerage costs being a percent is consistent with the fee schedule for large volume discount brokers. This results in a flat total cost function and implies a flat efficient frontier k% below the efficient frontier without transactions costs. A flat efficient frontier implies that a mutual fund should invest in the complete universe of securities.

Mutual funds charge a management fee of $X\%$ of assets. This fee is a reduction from the returns for individual investors. Consequently, the management fee shifts the efficient frontier for mutual fund investors downward by $X\%$. Thus, the efficient frontier with all costs included is the efficient frontier without any transactions costs minus $k\%$ minus $X\%$.

Figure 20 shows the efficient frontiers for direct investment by investors with different wealth levels. The wealthiest individuals have the highest efficient frontiers and the largest optimal portfolio sizes.

Since the wealthiest investors in Figure 20 have optimal portfolios lying above the mutual fund efficient frontier, these wealthy investors will invest directly. In contrast, investors with relatively low wealth will invest through a mutual fund since the mutual fund efficient frontier lies above their optimal portfolio from direct investment. In effect, the cost per dollar of direct investment is relatively high for small investors.

Mutual funds may charge different management fees. The higher the management fee, the lower the mutual fund efficient frontier and the fewer the number of investors in the mutual fund. Each mutual fund management firm must choose a management fee sufficient to cover the management company's cost, including a fair return. In a market with average securities, investors will choose between mutual funds on the basis of management fees. In equilibrium, there should be one unique management fee charged by all mutual funds. A firm charging a lower fee would drive every other firm out of business. Firms with a higher fee would have no investors. This unique equilibrium management fee implies a minimum size for a fund to cover the management company's expenses. Smaller-sized funds will find it unprofitable to operate because the management fee will not cover the fund's costs.

Figure 20 clearly shows that investor cost functions constrain mutual fund management fees. If the fees exceed investor costs for all investors, no investors would choose mutual funds. If the fees are very small, only the very largest investors would invest directly.

[12] Many funds charge a sales commission. But this fee is paid to those who sell the fund. It is essentially a marketing cost.

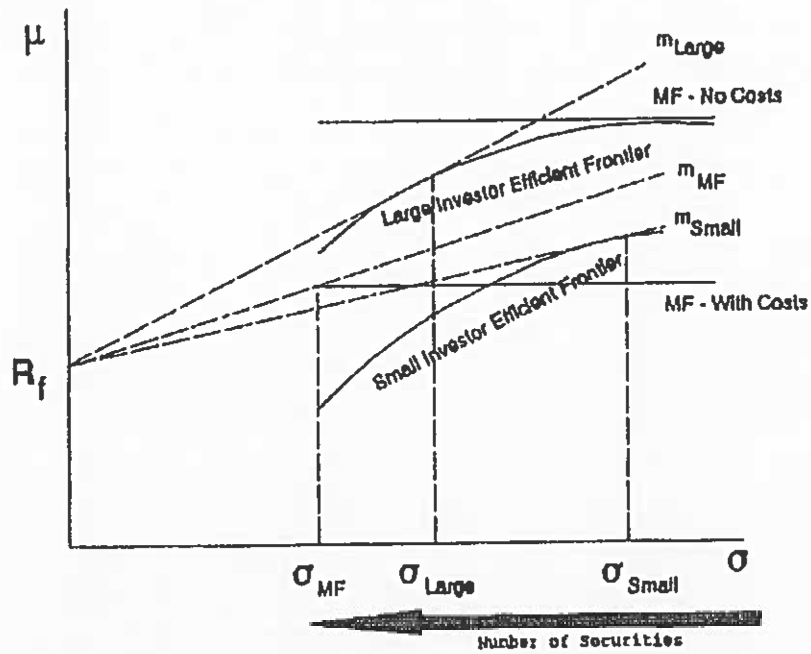

Figure 20: Mutual Funds Versus Direct Investment (assuming $\alpha = 1$)

VI. SUPERIOR INFORMATION AND THE OPTIMAL DIVERSIFICATION STRATEGY

Earlier sections examined optimal portfolio size and diversification gains when an investor chooses from a set of identically distributed assets representing the average investment opportunity in the market. In this section, a superior asset is introduced. The superior asset has a higher mean return and/or lower standard deviation and/or lower correlation with the individual assets. A superior asset can significantly reduce the gains from diversification and cause concentration of the portfolio.

ASSUMPTIONS

This paper is concerned with the optimization problem of individual investors in a mean-variance model. The paper is not a general equilibrium model such as the CAPM. This paper assumes some given set of information. Then, the individual investor constructs a portfolio according to the mean-variance model, whether or not the market is in general equilibrium. Conditions when investors will not diversify are derived. These conditions are consistent with the observed tendency of actual investors not to hold diversified portfolios.

If markets are weak form efficient, investors using a mean-variance model without access to inside information hold a well-diversified portfolio.[13] However, if the market is not strong form efficient (i.e., inside information is valuable), some investors will have access to valuable information and they will not diversify as shown below.

There is considerable evidence about the ability of some investors to beat the market. Black (1973) presents evidence that stocks recommended by Value Line Investment Survey outperform the market. Stickel (1985) claims that Value Line has private superior information not reflected in market prices. Thus, investors acting in accordance with Value Line's recommendations find it optimal to invest in a subset of assets, as we show below.

The implications of private information for market equilibrium are beyond the scope of this paper. The following papers derive market equilibrium where investors hold different portfolios in a mean-variance world.

1. Lintner (1969) derives equilibrium market prices in a world of asymmetric information. Each investor may construct a different portfolio. The market equilibrium is a weighted average of the expectations of different investors.
2. Levy (1978) proves that the market clears although each investor holds a portfolio that is a subset of the total number of securities in the market. In the Levy model, investors do not hold all securities in the market as a result of transactions costs, indivisibility of investment, and the cost of keeping track of new information on a large number of securities. All of these costs are related to the wealth of the investor. Levy assumes that investors do not hold the market portfolio. The results below provide an explanation why individual investors hold a small number of securities in their portfolio.
3. Merton (1987) develops a two-period model of capital market equilibrium in which each investor has information about a subset of the available securities in the market. Merton's results are similar to Levy's (1978). Both models generalize the CAPM by allowing the market to clear although investors do not hold the market portfolio.
4. Green and Hollifield (1993) show that variance minimization of the portfolio does not necessarily lead to a well-diversified portfolio because the portfolio is dominated by a single factor in the covariance structure of returns.

The Optimal Proportion Invested in the Superior Asset

An informed investor constructs a portfolio which may include both average assets and the superior asset. The informed investor has superior information that is not revealed to the rest of the market. The portfolio may include up to $n + 1$ assets:

[13] See Blume and Siegel (1992), p. 15.

n average assets and the superior asset. The rate of return for the total portfolio is composed of two parts: the superior asset and the aggregate portfolio of n average assets. The rate of return on an aggregate equally-weighted portfolio composed of n average assets is R_n. R_s (s stands for superior) is the rate of return on the superior asset. Denote by R_p the rate of return on the total portfolio. R_p is given by

$$R_p = pR_s + (1-p)R_n \tag{44}$$

where p is the proportion of wealth invested in the superior asset and $1-p$ is the proportion invested in the aggregate portfolio of average assets. The proportion invested in each of the average assets is $(1-p)/n$.

Investors can lend or borrow at a riskless interest rate, denoted by R_f. The objective function of the investor is to maximize the unit price of risk m, where

$$m = \frac{[\mu_p - R_f]}{\sigma_p} \tag{45}$$

- μ_p is the expected return on the portfolio.
- σ_p is the standard deviation of the portfolio.
- μ_s and μ_n are the expected returns of the superior asset and of the average (aggregate) asset portfolio, respectively.
- σ_s and σ_n are the standard deviation of the return on the superior asset and average asset portfolio, respectively.
- r_{sn} is the correlation coefficient between the superior asset and the portfolio of average assets.

Given this notation,

$$\mu_p = p\mu_s + (1-p)\mu_n \tag{46}$$

$$\sigma_p^2 = p^2\sigma_s^2 + (1-p)^2\sigma_n^2 + 2p(1-p)\sigma_s\sigma_n r_{sn} \tag{47}$$

The introduction of a superior asset alters the gains from diversification. To achieve the optimal portfolio, an investor should maximize the price of risk (m) with respect to p. The first order condition is derived by taking the derivative of m with respect to p and equating the derivative to zero. As shown in Appendix C, the optimal proportion p in the superior asset is

$$p = \frac{\mu - r_{sn}\sigma}{\mu + \sigma^2 - r_{sn}\sigma(1+\mu)} \tag{48}$$

where

$$\mu = \frac{(\mu_s - R_f)}{(\mu_n - R_f)} \tag{49}$$

$$\sigma = \frac{\sigma_s}{\sigma_n} \tag{50}$$

μ and σ are the ratios of risk premiums and standard deviations of the two assets. If short sales are not allowed, the optimal proportion in the superior asset is the minimum of p (with $1 - p$ invested in average assets) or 100% in the superior asset.

Define r_{sa} as the correlation coefficient between the superior asset and each of the average assets. Since[14] $r_{sn} = r_{sa}/(r^*)^{.5}$ and $\sigma_n = \sigma_a(r^*)^{.5}$, where $r^* = r + (1 - r)/n$, the expression for p may be rewritten as[15]

$$p = \frac{\mu - \frac{\sigma_s r_{sa}}{\sigma_a r^*}}{\mu + \frac{\sigma_s^2}{\sigma_a^2 r^*} - \frac{\sigma_s r_{sa}(1+\mu)}{\sigma_a r^*}} \tag{51}$$

The optimal diversification strategy is a function of the relative risk premium, μ, the standard deviations, σ_s and σ_a, and the correlation coefficients r_{sa} and r^*. The proportion invested in the superior asset will increase as μ increases and as σ_s decreases.[16] The impact of r_{sa} is more complicated.[17]

[14]Let R_s and R_n be the returns on the superior asset and the portfolio of n average assets, respectively. R is the return on individual average assets. COV denotes covariance. Since $1/n$ is the proportion invested in each of the average assets,

$$\begin{aligned} \text{COV}(R_s, R_n) &= \text{COV}(R_s, R/n) + \cdots + \text{COV}(R_s, R/n) \\ &= (\sigma_s r_a r_{sa})/n + \cdots + (\sigma_s r_a r_{sa})/n \\ &= \sigma_s \sigma_a r_{sa} \end{aligned}$$

Since

$$\begin{aligned} \text{COV}(R_s, R_n) &= \sigma_s \sigma_n r_{sn} \\ \sigma_n r_{sn} &= \sigma_a r_{sa} \\ r_{sn} &= (\sigma_a r_{sa})/\sigma_n \end{aligned}$$

Since

$$\begin{aligned} \sigma_n &= \sigma_n (r + (1 - r)/n)^{.5} \\ r_{sn} &= r_{sa}(r + (1 - r)/n)^{.5} \end{aligned}$$

[15]In the case when the number of average assets is infinite, $r^* = r$ and

$$p = \frac{\mu - \frac{\sigma_s r_{sa}}{\sigma_a r}}{\mu + \frac{\sigma_s^2}{\sigma_a^2 r} - \frac{\sigma_s r_{sa}(1+\mu)}{\sigma_a r}} .$$

[16]See Appendix C for proofs.

[17]$dp/dr_{sa} \gtreqless 0$ as $\mu \gtreqless \sigma_s/\sigma_a\sqrt{r^*}$. See Appendix D for proofs.

Conditions for a Superior Asset ($p > 0$)

If the number of securities in the optimal portfolio of average assets is infinite,[18] then the new asset is superior if $p > 0$. From equation (48),

$$p > 0 \text{ as } \mu > r_{sn}\sigma \tag{52}$$

$$\frac{\mu_s - R_f}{\sigma_s} > \left[\frac{\mu_n - R_f}{\sigma_n}\right] r_{sn} \tag{53}$$

Since

$$\sigma_n = \sigma_a\sqrt{r^*} \tag{54}$$

$$r_{sn} = r_{sa}\sigma_a/\sigma_n \tag{55}$$

and

$$\mu_a = \mu_n \tag{56}$$

This is equivalent to

$$\frac{\mu_s - R_f}{\sigma_s} > \left[\frac{\mu_a - R_f}{\sigma_a}\right] \frac{r_{sa}}{r^*} \tag{57}$$

This inequality can be rewritten as[19]

$$m_s > m_a\left[\frac{r_{sa}}{r^*}\right] \tag{58}$$

This has an interesting geometric interpretation in Figure 21. $[\mu_a - R_f]/\sigma$ is the slope of the ray going through the risk-free rate and the new asset. This ray is denoted by m_s. $[\mu_a - R_f]/\sigma_a$ is the slope of the ray between the risk-free and an individual average asset and is denoted by m_a. Multiplying m_a by r_{sa}/r^* pivots the line m_a. The line pivots downward (upward) if r_{sa} is less (more) than r^*. A new asset should be included in the portfolio as long as m_s lies above $m_a r_{sa}/r^*$.

Conditions for Specialization ($p \geq 1$)

As p increases, the investor will begin to completely specialize (i.e., $p \geq 1$) the portfolio in the superior asset at some point. If this happens, the benefits of the

[18] If the number of securities in the optimal portfolio of average assets is finite and equal to n, the new asset is superior to the average assets if it has a larger than proportional share, i.e., if $p > 1/(n+1)$. Algebraically, this occurs if

$$\mu > \frac{\sigma(\sigma + nr_{sn})}{n + \sigma r_{sn}}$$

[19] If n is infinite, $r^* = r$ and we have

$$m_s > m_a r_{sa}/r$$

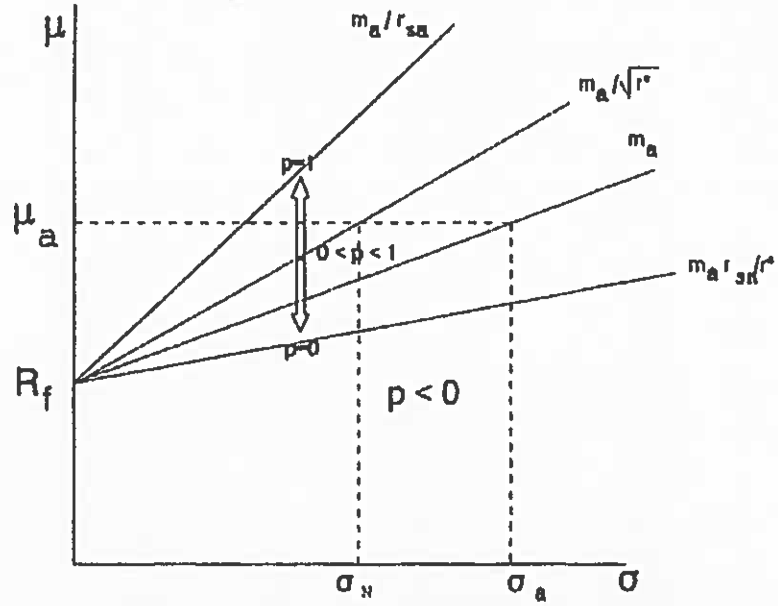

Figure 21: Conditions for Including a Superior Asset

superior asset have outweighed the advantage of widespread diversification. When short sales are excluded, the investor will begin to completely specialize and hold 100% of wealth in asset s if $p > 1$. As shown in Appendix C,

$$p \geq 1 \text{ as } r_{sn} \geq \frac{\sigma}{\mu} \tag{59}$$

Since $\sigma \geq 0$ and $\mu \geq 0$, $\sigma/\mu \geq 0$. Thus, complete specialization ($p \geq 1$) will never occur if the correlation coefficient r_{sn} is negative. Since $r_{sn} = r_{sa}/(r^*)^{.5}$ and $r^* \geq 0$, complete specialization will not occur if the superior asset is negatively correlated with the average assets. This result holds even if the mean return on the superior asset is very high or its standard deviation is very low. The reason is that the diversification benefits of a negative correlation coefficient outweigh the higher mean.

Substituting the definitions of μ and σ and rearranging, complete specialization in the superior asset occurs if

$$\left[\frac{\mu_s - R_f}{\sigma_s}\right] r_{sn} \geq \frac{\mu_n - R_f}{\sigma_n} \tag{60}$$

Or equivalently (for $r_{sa} > 0$) if[20]

$$\frac{\mu_s - R_f}{\sigma_s} \geq \frac{\mu_s - R_f}{\sigma_a r_{sa}} \tag{61}$$

$$m_s \geq \frac{m_a}{r_{sa}} \tag{62}$$

This result has an interesting geometric interpretation shown in Figure 21. The term m_s is the excess return per unit of risk and is often called the price of risk for the superior asset. m_a is the price of risk for the average asset. Dividing m_a by r_{sa} in rotates the line m_a upward. If the superior asset is perfectly correlated with the individual average assets (i.e. $r_{sa} = 1$) specialization occurs if $m_s \geq m_a$. Complete specialization is less likely as this correlation coefficient (r_{sa}) declines because the benefits of diversification are greater. Specialization is more likely as the mean return on the superior asset (μ_s) increases and as its standard deviation (σ_s) decreases.

The Amount of Information Required for Specialization

The preceding results showed that specialization in a superior asset is possible. The amount of incremental information needed to cause specialization is now addressed. Under quite plausible assumptions, the amount of incremental information for specialization is shown to be quite small. Specialization is a realistic possibility for investors with superior information.

Define the amount of incremental mean information for complete specialization in the superior asset as $\mu_s - \mu_a$, the mean return of the superior asset minus the mean of the average assets. By solving equation (61) for μ_s, subtracting μ_a from both sides, and rearranging,

$$\mu_s - \mu_a \geq (\mu_a - R_f)\left[\frac{\sigma_s}{\sigma_a r_{sa}} - 1\right] \tag{63}$$

Whenever the mean return on the superior asset equals or exceeds the mean return on average assets (μ_a) by $[\mu_a - R_f][\mu_s/\sigma_a r_{sa} - 1]$, complete specialization in the superior asset occurs. Notice that specialization is more likely (1) if the standard deviation of the superior asset is small relative to the standard deviation of average assets (i.e. σ_s/σ_a is small) or (2) if the correlation coefficient (r_{sa}) between superior and average assets is high.

Table 9 shows the values of $\sigma_s/\sigma_a r_{sa} - 1$ for a wide range of values of r_{sa} and σ_s/σ_a. First, if σ_s/σ_a equals r_{sa}, $\sigma_s/\sigma_a r_{sa} - 1$ equals zero; this is the main diagonal in the table; specialization occurs if μ_s equals or exceeds μ_a. Second, the case of $\sigma_s/\sigma_a < r_{sa}$ lies above the main diagonal. Specialization occurs if $\mu_s - \mu_a$

[20] Notice that complete specialization is independent of n.

exceeds the number in the table times $\mu_a - R_f$. Since the number in the table is negative above the main diagonal, specialization clearly occurs if $\mu_s > \mu_a$. Third, if $\sigma_s/\sigma_a > r_{sa}$, $\sigma_s/\sigma_a r_{sa} - 1$ is positive; this case is below the main diagonal. Specialization occurs if $\mu_s - \mu_a$ exceeds the number in the table times $\mu_a - R_f$.

In the case in which $\sigma_s = \sigma_a$, equation (63) simplifies to

$$\mu_s - \mu_a = (\mu_a - R_f)\left[\frac{1}{r_{sa}} - 1\right] \tag{65}$$

If $\sigma_s < \sigma_a$, this equation provides an upper bound for the amount of incremental information necessary for complete specialization. This bound is the bottom row of Table 9.

For many securities, the correlation coefficient with other securities is between .50 and 1.0. If r_{sa} equals .50, equation (63) shows that the incremental information for complete specialization is $(\mu_a - R_f)(2\sigma_s/\sigma_a - 1)$. If r_{sa} equals 1.0, $\mu_s - \mu_a$ equals $(\mu_s - R_f)(\sigma_s/\sigma_a - 1)$. Consequently, for many securities the amount of incremental information lies between $(\mu_a - R_f)(2\sigma_s/\sigma_a - 1)$ and $(\mu_a - R_f)/(\sigma_s/\sigma_a - 1)$.

In Table 9, consider the segment for which $.50 \leq r < 1.0$ and $.50 < \sigma_s/\sigma_a \leq 1$. The amount of special information required for complete specialization is quite small, especially when close to the main diagonal (i.e. $\sigma_s/\sigma_a = r_{sa}$). For example, when $r_{sa} = .90$ and $\sigma_s/\sigma_a = 1$, $\sigma_s/\sigma_a r_{sa} - 1 = .11$. The incremental mean information required for complete specialization is 11% of $\mu_a - R_f$. For $r_{sa} = .80$, and $\sigma_s/\sigma_a = .90$, it is 13%, and so on.

Table 9 shows that surprisingly small amounts of incremental mean information are required for complete specialization assuming that r_{sa} is .50 or more. If $r_{sa} \geq .50$ and $\sigma_s/\sigma_a \leq 1$, the maximum value of $\sigma_s/\sigma_a r_{sa} - 1$ is 1. For many values of r_{sa} and σ_s/σ_a, $\sigma_s/\sigma_a r_{sa} - 1$ is considerably less than 1.0, implying that the incremental mean information for complete specialization is considerably less than $\mu_a - R_f$.

VII. TRANSACTION COSTS AND A SUPERIOR ASSET

The introduction of transactions costs was shown earlier to reduce the number of average securities held in an optimal portfolio in the case where transactions costs are proportional to the percentage invested in risky assets. Without transactions costs, the investor optimally holds as many securities as possible. With transactions costs, the optimal number of average securities is reduced to n. See Table 10.

In the case of a superior asset and zero transactions costs, a superior asset was shown earlier to cause a proportion of the portfolio to be invested in the superior asset. However, if a part of the optimal portfolio includes average assets, the investor holds as many average securities as possible. See Table 10.

If transactions costs *and* a superior asset exist, both factors generally act in the same direction. That is, a proportion of the portfolio is held in the superior asset and the rest of the portfolio is held in average assets. Usually, the number

Table 9: $\sigma_s/\sigma_a r_{sa} - 1$
The Amount of Incremental Information for Complete Specialization as a Percent of $\mu_a - R_f$

	Correlation Coefficient r_{sa}									
Ratio σ_s/σ_a	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.0
.10	0	−.50	−.67	−.75	−.80	−.83	−.86	−.88	−.89	−.90
.20	1	0	−.33	−.50	−.60	−.67	−.71	−.75	−.78	−.80
.30	2.0	.50	0	−.25	−.40	−.50	−.57	−.63	−.67	−.70
.40	3.0	1.00	.33	0	−.20	−.33	−.43	−.50	−.56	−.60
.50	4.0	1.50	.67	.25	0	−.17	−.29	−.38	−.44	−.50
.60	5.0	2.00	1.00	.50	.20	0	−.14	−.25	−.33	−.40
.70	6.0	2.50	1.33	.75	.40	.17	0	−.13	−.22	−.30
.80	7.0	3.00	1.67	1.00	.60	.33	.14	0	−.11	−.20
.90	8.0	3.50	2.00	1.25	.80	.50	.29	.13	0	−.10
1.00	9.0	4.00	2.33	1.50	1.00	.67	.43	.25	.11	0

Note: The amount of information for complete specialization is

$$[\mu_a - R_f]\left[\frac{\sigma_s}{\sigma_a r_{sa}} - 1\right] \quad (64)$$

Table 10: Number of Average Securities in Optimal Portfolio

	No Transactions Costs	Positive Transactions Costs
Average Assets Only	All Average Assets	n
Average and Superior Assets	All Average Assets	n_s

N = number of average assets
n = optimal number of average assets with transactions costs
n_s = optimal number of average assets with transactions costs and superior information
$n < N$
$n_s < N$

of average assets with transactions costs and a superior asset (n_s) is less than the number without the superior asset (n). See Table 10.

The introduction of a superior asset to a world of average assets with transactions costs has two impacts.[21] First, since some proportion of the portfolio is invested in the superior asset, a smaller dollar amount is invested in average assets. From the discussion of transactions costs and portfolio size, a smaller amount of wealth invested in average assets increases the cost per dollar if there are wealth economies and reduces the optimal number of average assets in the portfolio. Second, since there is an additional (superior) asset, there is a potential for an added diversification benefit from increasing the number of average assets. The optimal number of average assets depends upon the net impact of both of these effects.

Extensive simulations have found that the number of average assets is reduced for linear total cost functions (assuming transactions costs are proportional to α), as long as the correlation coefficient (r_{sa}) between the superior asset and average assets is non-negative. If the total cost function is convex (i.e. $dC/dn > 0$ and $dC^2/dn^2 > 0$), the cost *dis*advantage of adding securities is larger, implying that the number of average assets must also decrease for $r_{sa} > 0$ as the superior asset is added.

For negative correlation coefficients, a superior asset may actually increase the number of average assets, because the diversification effect outweighs the wealth effect. For example, consider a linear total cost function with $a/W = .0005$, fixed

[21] A formal analysis is presented in Appendix B.

cost of zero, $\mu_s = .10$, $\mu_a = .08$, $R_f = .03$, $\sigma_s = \sigma_a = .02$, $r = .50$, and $r_{sa} = -.50$. Without a superior asset, the optimal portfolio has six average assets (n). With a superior asset, the optimal portfolio has eight superior assets ($n_s = 8$ and $p = .43$).

In the usual case, the wealth effect is stronger than the diversification effect and the introduction of a superior asset reduces the optimal number of average assets.

VIII. CONCLUDING REMARKS

The potential gains from portfolio diversification are well known. However, many practitioners refrain from holding a portfolio with a large number of securities. Using a mean-variance framework, this paper attempts to resolve this apparent contradiction by comparing the incremental gains from diversification against two offsetting factors: transaction costs and superior information.

The main results of the paper are:

1. About 90% of potential diversification benefits are shown to occur with a portfolio size of 10 securities for a correlation coefficient of .50 or larger if all securities have the same covariances. If covariances differ, somewhat larger portfolios are required.
2. In the presence of transaction costs, portfolio separation does not hold. Optimal portfolio size depends upon the transactions cost function, wealth economies, and investor preferences. If brokers charge a constant dollar charge per share purchased, investors hold all available securities. If the brokerage cost function is linear with a charge for each stock purchased, the optimal portfolio size decreases as the correlation coefficient between securities increases, as wealth decreases, and as brokerage costs decrease. For many small investors, optimal portfolio size is quite small for available brokerage schedules.
3. The existence of a superior asset brings about a significant reduction in the gains from diversification for relatively small amounts of special information.
4. In general, the existence of both a superior asset and transactions costs reduce the optimal number of average assets in the portfolio. In this world of concentrated portfolios, the Generalized CAPM of Levy (1978) describes equilibrium.

IX. REFERENCES

Ben-Horim, Moshe and Haim Levy. 1980. "Total Risk, Diversifiable Risk and Non-Diversifiable Risk: A Pedagogic Note." *Journal of Financial and Quantitative Analysis* 15: 289–297.

Best, Michael J. and Robert R. Grauer. 1991. "On the Sensitivity of MeanVariance-Efficient Portfolios to Changes in Asset Means: Some Analytical and Computational Results."

Review of Financial Studies 4: 315–342.

Bird, Ron and Mark Tippett. 1986. "Naive Diversification and Portfolio Risk—A Note." *Management Science* 32: 244–51.

Black, F. 1973. "Yes, Virginia, There is Hope: Tests of the Value Line Ranking System." *Financial Analysts Journal* 29: 10–14.

Blume, Marshall E. and Irwin Friend. 1975. "The Asset Structure of Individual Portfolios and Some Implications for Utility Functions." *Journal of Finance* 30: 585–603.

Blume, Marshall E. and Irwin Friend. 1978. *The Changing Role of the Individual Investor*. New York, John Wiley & Sons.

Blume, Marshall E., Jean Crockett, and Irwin Friend. 1974. "Stock Ownership in the United States: Characteristics and Trends." *Survey of Current Business* 54: 16–40.

Blume, M. E. and J. J. Siegel. 1992. "The Theory of Security Pricing and Market Structure." *Financial Markets, Institutions, and Instruments* 1:3.

Brennan, M. J. 1975. "The Optimal Number of Securities in a Risky Asset Portfolio When There Are Fixed Costs of Transactions: Theory and Some Empirical Results." *Journal of Financial and Quantitative Analysis* 10: 483–496.

Brumelle, Shelby L. 1974. "When Does Diversification Between Two Investments Pay?" *Journal of Financial and Quantitative Analysis* 9: 231–40.

Conine, Thomas E. and Maurry J. Tamarkin. 1981. "On Diversification Given Asymmetry in Returns." *Journal of Finance* 36: 1143–1155.

Elton, Edwin J. and Martin J. Gruber. 1977. "Risk Reduction and Portfolio Size: An Analytical Solution." *The Journal of Business* 50: 415–437.

Evans, John L. and Stephen A. Archer. 1968. "Diversification and the Reduction of Dispersion: An Empirical Analysis." *Journal of Financial and Quantitative Analysis* 23: 761–767.

French, Kenneth and James Poterba. 1986. "Investor Diversification and International Equity Markets." *American Economic Review* 81: 222–226.

Goldsmith, David. 1976. "Transactions Costs and the Theory of Portfolio Selection." *Journal of Finance* 36: 1127–1139.

Green, R. C. and B. Hollifield. 1992. "When Will Mean-Variance Efficient Portfolios be Well Diversified?" *Journal of Finance* 47: 1785–1810.

Hadar, J. and W. R. Russell. 1971. "Stochastic Dominance and Diversification." *Journal of Economic Theory* 3: 288–305.

Hadar, J. and W. R. Russell. 1974. "Diversification of Interdependent Prospects." *Journal of Economic Theory* 7: 231–240.

Jacob, Nancy L. 1984. "A Limited-Diversification Portfolio Selection Model for the Small Investor." *Journal of Finance* 29: 847–856.

Johnson, K. H. and D. S. Shannon. 1974. "A Note on Diversification and the Reduction of Dispersion." *Journal of Financial Economics* 1: 365–372.

King, Benjamin. 1966. "Market and Industry Factors in Stock Price Behavior." *Journal of Business* 39: 139–190.

Kroll, Yoram, Haim Levy, and Harry Markowitz. 1984. "Mean Variance Versus Direct Utility Maximization." *Journal of Finance* 39: 47–61.

Latane, Henry A. and William E. Young. 1969. "Test of Portfolio Building Rules." *Journal of Finance* 24: 595–612.

Lease, Ronald C., William G. Lewellen, and Gary G. Schlarbaum. 1976. "Market Segmentation: Evidence on the Individual Investor." *Financial Analysts Journal* 32: 53–60.

Levy, Haim. 1973. "The Demand for Assets Under Conditions of Risk." *Journal of Finance* 28: 79–96.

Levy, Haim. 1978. "Equilibrium in an Imperfect Market: A Constraint on the Number of Securities in the Portfolio." *American Economic Review* 68: 643–658.

Levy, Haim and Harry Markowitz. 1979. "Approximating Utility by a Function of Mean and Variance." *American Economic Review* 69: 308–17.

Levy, Haim and Marshall Sarnat. 1984. *Portfolio and Investment Selection: Theory and Practice*. Englewood Cliffs, NY: Prentice Hall.

Levy, Haim. 1979. "Does Diversification Always Pay?" *TIMS Studies in the Management Science* II: 86–94.

Lintner, J. 1969. "The Aggregation of Investors' Diverse Judgments and Preferences in Purely Competitive Security Markets." *Journal of Financial and Quantitative Analysis* 4: 347–400.

Livingston, Miles. 1977. "Industry Movements of Common Stocks." *Journal of Finance* 32: 861–874.

MacMinn, Richard D. 1984. "A General Diversification Theorem: A Note." *Journal of Finance* 39: 541–550.

Mao, James C. T. 1970. "Essentials of Portfolio Diversification Strategy." *Journal of Finance* 15: 1109–1121.

Markowitz, Harry M. 1959. *Portfolio Selection: Efficient Diversification of Investment*. New York.

Markowitz, Harry M. 1952. "Portfolio Selection." *Journal of Finance* 7: 77–91.

Merton, Robert. 1987. "A Simple Model of Capital market Equilibrium with Incomplete Information." *Journal of Finance* 42: 483–510.

Meyers, Steven L. 1973. "A Re-Examination of Market and Industry Factors in Stock Price Behavior." *Journal of Finance* 28: 695–705.

Mokkelbost, Per B. 1971. "Unsystematic Risk Over Time." *Journal of Financial and Quantitative Analysis* 6: 78596.

Pulley, Lawrence. 1981. "A General Mean-Variance Approximation to Expected-Utility for Short Holding Periods." *Journal of Financial and Quantitative Analysis* 16: 361–73.

Pulley, Lawrence. 1983. "Mean-Variance Approximations to Expected Logarithmic Utility." *Operations Research* 31: 685–96.

Pulley, Lawrence. 1985. "Mean-Variance Versus Direct Utility Maximization: A Comment." *Journal of Finance* 40: 601–602.

Samuelson, Paul S. 1967. "General Proof that Diversification Pays." *Journal of Financial and Quantitative Analysis* 2: 1–13.

Sharpe, William F. 1972. "Risk, Market Sensitivity and Diversification." *Financial Analysts Journal* 28: 74–79.

Simkowitz, Michael A. and William L. Beedles. 1978. "Diversification in a Three-Moment World." *Journal of Financial and Quantitative Analysis* 13: 927–941.

Smith, Keith V. and John C. Schreiner. 1980. "Direct vs. Indirect Diversification." *Financial Analysts Journal* 26: 33–38.

Solnik, Bruno H. 1974. "Why Not Diversify Internationally Rather than Domestically." *Financial Analysts Journal* 30: 48–54.

Stickel, S. E. 1985. "The Effect of Value Line Investment Survey Rank Changes on Common Stock Prices." *Journal of Financial Economics* 14: 121–143.

Tole, Thomas M. 1982. "You Can't Diversify Without Diversifying." *Journal of Portfolio Management* 8: 5–11.

Upson, Roger B., Paul F. Jessup and Keishiro Matsumoto. 1975. "Portfolio Diversification Strategies." *Financial Analysts Journal* 31: 86–88.

Wagner, W. H. and S. C. Lau. 1971. "The Effect of Diversification on Risk." *Financial Analysts Journal* 27: 48–53.

X. APPENDICES

Appendix A: A Symmetric Distribution of Standard Deviations

The text of this paper assumes that all securities have the same standard deviations. Now consider the case where the standard deviations differ, but are *symmetric* about their mean σ. The correlations are identical. There are several cases to examine.

Case 1. There are two values of the standard deviation: $\sigma - \Delta$ and $\sigma + \Delta$.

Case 2. There are three values of the standard deviation: $\sigma - \Delta, \sigma$, and $\sigma + \Delta$.

Case 3. There are four values of the standard deviation: $\sigma - \Delta, \sigma - \Delta/2, \sigma + \Delta/2, \sigma + \Delta$.

Case 4. There are five values of the standard deviation: $\sigma - \Delta, \sigma - \Delta/2, \sigma + \Delta/2$, and $\sigma + \Delta$.

etc.

Assume that an investor follows the strategy of putting an equal proportion of wealth in each level of standard deviation. In Case 1, the investor buys $n/2$ securities of each of the 2 types. In case 2, the investor buys $n/3$ in each of the 3 types. And so on.

It is obvious that the portfolio standard deviation in Case 1 must be larger than the portfolio standard deviation in each of the other cases. Thus, Case 1 is an upper bound for portfolio standard deviation.

At the end of appendix A, the portfolio standard deviation in Case 1 is shown to equal

$$\sigma_p = \sigma\sqrt{r + \left[\frac{1-r}{n}\right]\left[1 + \frac{\Delta^2}{\sigma^2}\right]}$$

The percentage of attainable risk reduction is

$$x = 1 - \frac{\sqrt{r + \left[\frac{1-r}{n}\right]\left[1 + \frac{\Delta^2}{\sigma^2}\right]}}{1 - \sqrt{r}}$$

Solve for the portfolio size n for a given level of risk reduction x.

$$n = \frac{(1-r)\left[1 + \frac{\Delta^2}{\sigma^2}\right]}{\left[1 - x(1 - \sqrt{r})\right]^2 - r}$$

When the standard deviations are the same, $\Delta = 0$. Call the value of n for which $\Delta = 0$, n_0. Then,

$$n = n_0\left[1 + \frac{\Delta^2}{\sigma^2}\right]$$

The highest possible value of Δ is σ, implying that $\Delta^2/\sigma^2 = 1$ and

$$n \leq 2n_0$$

Thus, for a symmetrical distribution of the standard deviation about its mean, the portfolio size n to achieve a given percentage of risk reduction can never exceed twice the value in Table 1. For example, for $\Delta = 0$, approximately 88% of attainable risk reduction can be achieved with 10 different securities if $r \geq .50$. When $\Delta > 0$, no more than 20 securities are required for the same 88%.

The preceding argument assumes that the distribution of standard deviations is symmetric about its mean. If the symmetry assumption is suspended, the same upper bound applies if the distribution is skewed to the right and the largest standard deviation does not exceed twice the mean.

If the distribution of standard deviations is nonsymmetric about the mean standard deviation, let the largest standard deviation equal $\sigma + k\sigma$. In other words, $\Delta = k\sigma$. Then, by the same line of reasoning, the largest value of the optimal portfolio size cannot exceed $n_0(1 + k\sigma/\sigma) = n_0(1 + k)$. Optimal portfolio sizes cannot exceed $(1 + k)$ times the numbers in Table 1.

The preceding arguments assume equal proportions invested in each security to derive upper bounds for portfolio size. In general, putting equal proportions in each security is an inferior strategy when standard deviations differ. It is better to concentrate the portfolio in the lower standard deviation securities, ceteris paribus. Consequently, fewer securities than the upper bounds are required to attain a given level of portfolio risk.

PROPOSITION. *In Case I,*

$$\sigma_p^2 = \sigma^2\left\{r + \left[\frac{1-r}{n}\right]\left[1 + \frac{\Delta^2}{\sigma^2}\right]\right\}$$

The two standard deviations in Case 1 are $\sigma - \Delta$ and $\sigma + \Delta$. Choose $n/2$ from each group.

The variance covariance matrix is

Quadrant I	Quadrant II
$(\sigma - \Delta)(\sigma - \Delta)r \ldots$ $(\sigma - \Delta)r(\sigma - \Delta)$ . . .	$(\sigma^2 - \Delta^2)r \ldots$ . . . . .
$(\sigma^2 - \Delta^2)r \ldots$	$(\sigma + \Delta(\sigma + \Delta)r \ldots$ $(\sigma + \Delta)r(\sigma + \Delta)$ . . . .

The sums of four quadrants are

Quadrant I.

$$\frac{n}{2}[\sigma^2 - 2\sigma\Delta + \Delta^2] + \frac{n}{2}\left[\frac{n}{2} - 1\right][\sigma - \Delta]^2 r$$

Quadrant II.

$$\left[\frac{n}{2}\right]\left[\frac{n}{2}\right][\sigma^2 - \Delta^2]r$$

Quadrant III.

$$\left[\frac{n}{2}\right]\left[\frac{n}{2}\right][\sigma^2 - \Delta^2]r$$

Quadrant IV. Take the sum of the terms and divide by n^2, the proportion invested in each security to get the variance.

$$\frac{n}{2}[\sigma^2 + 2\sigma\Delta + \Delta^2] + \frac{n}{2}\left[\frac{n}{2} - 1\right][\sigma + \Delta]^2 r$$

Appendix B: Derivation of dm/dn for Costs Proportional to α.

Algebraically, an investor would like to maximize the slope, m, of the rays through the risk-free rate and the various possible portfolios. This slope represents the market price of risk.

$$m = \frac{1}{\sigma_a}\left[\frac{\mu_a - R_f - C(n)}{\sqrt{r + \frac{1-r}{n}}}\right]$$

$$\frac{dm}{dn} = \frac{1}{\sigma_a}\left[\frac{\frac{-dC(n)}{dn}}{\sqrt{r+\frac{1-r}{n}}} + \frac{\frac{(\mu_a - R_f - C(n))(1-r)}{2n^2}}{\left[r+\frac{1-r}{n}\right]\sqrt{r+\frac{1-r}{n}}}\right]$$

Since $1/\sigma_a$ is in both terms, the optimal portfolio size does not depend upon the standard deviation.

$$\frac{dm}{dn} = \frac{\frac{[\mu_a - R_f C(n)][1-r]}{2n^2[r+(1-r)/n]} - \frac{dC(n)}{dn}}{\sigma_a\sqrt{r+(1-r)/n}}$$

Since

$$\frac{1-r}{2n^2[r+(1-r)/n]} = \frac{\frac{-d\sigma_n}{dn}}{\sigma_n}$$

$$\frac{dm}{dn} = \frac{[\mu_a - R_f - C(n)]\left[\frac{-d\sigma_n/dn}{\sigma_n}\right] - \frac{dC(n)}{dn}}{\sigma_n}$$

Therefore,

$$\frac{dm}{dn} \gtreqless 0 \quad \text{as} \quad \frac{\mu_a - R_f - C(n)}{\sigma_a} \gtreqless \frac{\frac{dC(n)}{dn}}{\frac{-d\sigma_n}{dn}}$$

$$\frac{dm}{dn} \gtreqless 0 \quad \text{as} \quad \begin{bmatrix}\text{excess return}\\ \text{per unit of}\\ \text{portfolio } \sigma\end{bmatrix} \gtreqless \frac{\Delta \text{ cost}}{-\Delta\sigma_n}$$

Optimal Portfolio Size with Linear Cost Function

The linear cost function is

$$C(n) = \frac{an}{W} + b$$

$$\frac{dC(n)}{dn} = \frac{a}{W}$$

Set dm/dn equal to zero.

$$\frac{[1-r]\left[\mu_a - R_f - \left(\frac{an}{W}+b\right)\right]}{2n^2\left[r+\frac{(1-r)}{n}\right]} = \frac{dC(n)}{dn}$$

Substitute for $dC(n)/dn$ and rearrange.

$$n^2\left[\frac{2r}{1-r}\right] + n[3] + [(-1)(\mu_a - R_f - b)]\left[\frac{W}{a}\right] = 0$$

If the correlation coefficient r is zero, this simplifies to

$$3n + [(-1)(\mu_a - R_f - b)]\left[\frac{W}{a}\right] = 0.$$

When $r = 0$, the optimal portfolio size is

$$n = \frac{\mu_a - R_f - b}{3a/W}$$

When r is not equal to zero, use the quadratic formula.

$$n = \frac{-3 + \sqrt{9 + \left[\frac{8r}{1-r}\right]\left[\frac{\mu_a - R_f - b}{a/W}\right]}}{\frac{4r}{1-r}}$$

PROOF THAT $dn/dr < 0$.

Express the optimal n as $NUM\ /\ DEN$. Then,

$$\frac{dn}{dr} = \left[-\frac{NUM}{DEN^2}\right]\left[\frac{4}{1-r} + \frac{4r}{(1-r)^2}\right] + \left[\frac{1}{DEN}\right]\left[\frac{.5}{\sqrt{X}}\right]\left[\frac{8W(\mu_a - R_f - b)}{a}\right]\left[\frac{1}{1-r} + \frac{r}{(1-r)^2}\right]$$

where

$$X = 9 + \left[\frac{8r}{1-r}\right]\left[\frac{W(\mu_a - R_f - b)}{a}\right]$$

$$\frac{dn}{dr} < 0 \text{ as}$$

$$\left[\frac{1}{2\sqrt{X}}\right]\left[\frac{8W(\mu_a - R_f - b)}{a(1-r)}\right]\left[1 + \frac{r}{1-r}\right] < \left[\frac{NUM}{DEN}\right]\left[\frac{4}{1-r}\right]\left[1 + \frac{r}{1-r}\right]$$

Cancel terms, substitute for $NUM\ /\ DEN$.

$$3\sqrt{X} < X - \left[\frac{4Wr}{1-r}\right]\left[\frac{\mu_a - R_f - b}{a}\right]$$

Substitute for X, and simplify the right side. Then square both sides.

$$81 + \left[\frac{72rW}{1-r}\right]\left[\frac{\mu_a - R_f - b}{a}\right] < \left[\frac{72rW}{1-r}\right]\left[\frac{\mu_a - R_f - b}{a}\right] + \left[\frac{16W^2r^2}{a^2(1-r)^2}\right][\mu_a - R_f - b]^2$$

Simplify.

$$0 < \left[\frac{16W^2r^2}{a^2(1-r)^2}\right][\mu_a - R_f - b]^2$$

Since the right side is positive, this condition must always hold.

Appendix C: Optimal Proportion Invested in Superior Asset

$$m_p = \frac{\mu_p - R_f}{\sigma_p}$$

$$m_p = \frac{p\mu_s + (1-p)\mu_n - R_f}{\sqrt{p^2\sigma_s^2 + (1-p)^2\sigma_n^2 + 2p(1-p)\sigma_s\sigma_n r_{sn}}}$$

Rewrite as

$$m_p = \frac{pA + B}{\sqrt{p^2 D - 2pE + \sigma_n^2}}$$

where

$$\begin{aligned} A &= \mu_s - \mu_n \\ B &= \mu_n - R_f \\ D &= \sigma_s^2 + \sigma_n^2 - 2\sigma_s\sigma_n r_{sn} \\ E &= \sigma_n^2 - \sigma_s\sigma_n r_{sn} \end{aligned}$$

$$\frac{dm_p}{dp} = \frac{A}{\sigma_p} - \frac{(pA+B)(1/2)(2pD - 2E)}{\sigma_p^2\sigma_p}$$

set $dm_p/dp = 0$

$$\begin{aligned} \frac{A}{\sigma_p} &= \frac{(pA+B)(pD-E)}{\sigma_p\sigma_p^2} \\ A\sigma_p^2 &= (pA+B)(pD-E) \end{aligned}$$

Substitute for σ_p^2

$$p^2AD - 2pAE + A\sigma_n^2 = p^2AD - pAE + pBD - BE$$

Solve for p

$$p = \frac{A\sigma_n^2 + BE}{AE + BD}$$

Substitute for A, B, D, E.

$$p = \frac{(\mu_s - \mu_n)\sigma_n^2 + (\mu_n - R_f)(\sigma_n^2 - \sigma_n\sigma_s r_{sn})}{(\mu_s - \mu_n)[\sigma_n^2 - \sigma_n\sigma_s r_{sn}] + (\mu_n - R_f)(\sigma_s^2 + \sigma_n^2 - 2\sigma_n\sigma_s r_{sn})}$$

Simplify the numerator and denominator and then divide each by $(\mu_n - R_f)\sigma_n^2$

$$p = \frac{\frac{\mu_s - R_f}{\mu_n - R_f} - \frac{\sigma_s}{\sigma_n} r_{sn}}{\frac{\mu_s - R_f}{\mu_n - R_f} + \frac{\sigma_s^2}{\sigma_n^2} - \frac{r_{sn}\sigma_s}{\sigma_n}\left[1 + \frac{\mu_s - R_f}{\mu_n - R_f}\right]}$$

$$p = \frac{\mu - \sigma r_{sn}}{\mu + \sigma^2 - \sigma r_{sn}(1 + \mu)}$$

For

$$\mu + \sigma^2 - \sigma r_{sn}(1 + \mu) > 0,$$

$p > 1$ when

$$\mu - \sigma r_{sn} > \mu + \sigma^2 - \sigma r_{sn}(1 + \mu)$$

$$\sigma r_{sn} \mu > \sigma^2$$

$$r_{sn} > \frac{\sigma}{\mu}$$

Appendix D: Superior Asset and No Transactions Costs

I. Proof that $dp/du > 0$ if $r_{sa} \leq (r^*)^{.5}$.

$$p = \frac{\mu - \frac{\sigma_s r_{sa}}{\sigma_a r^*}}{\mu + \frac{\sigma_s^2}{\sigma_a^2 r^*} - \frac{\sigma_s r_{sa}(1+\mu)}{\sigma_a r^*}} = \frac{NUM}{DEN}$$

$$\frac{dp}{du} = \frac{1}{DEN} - \frac{NUM}{DEN^2}\left[1 - \frac{\sigma_s r_{sa}}{\sigma_a r^*}\right]$$

Assuming $DEN > 0$,

$$\frac{dp}{du} \gtreqless 0 \quad \text{as} \quad 1 \gtreqless p\left[1 - \frac{\sigma_s r_{sa}}{\sigma_a r^*}\right]$$

Substitute for p, cross-multiplicity, cancel terms, resulting in

$$\frac{dp}{du} \gtreqless 0 \quad \text{as} \quad \frac{\sigma_s^2}{\sigma_s^2 r^*} \gtreqless \frac{\sigma_s^2 r_{sa}^2}{\sigma_a^2 r^{*2}}$$

$$\frac{dp}{du} \gtreqless 0 \quad \text{as} \quad \sqrt{r^*} \gtreqless r_{sa}$$

Assuming $r_{sn} \leq 1$, $r_{sa} \leq (r^*)^{.5}$, implying that $dp/du > 0$.

II. Proof that $dp/d\sigma_s < 0$

$$\frac{dp}{d\sigma_s} = \frac{\frac{-r_{sa}}{\sigma_a r^*}}{DEN} - \frac{NUM}{DEN^2}\left[\frac{2\sigma_s}{\sigma_a r^*} - \frac{r_{sa}(1 + \mu)}{\sigma_a r^*}\right]$$

Assuming $DEN > 0$,

$$\frac{dp}{d\sigma_s} \gtreqless 0 \quad \text{as} \quad p\left[\frac{r_{sa}(1+\mu)}{\sigma_a r^*} - \frac{2\sigma_s}{\sigma_a r^*}\right] \gtreqless \frac{r_{sa}}{\sigma_a r^*}$$

For $p > 0$,

$$\frac{dp}{d\sigma_s} \gtreqless 0 \quad \text{as} \quad r_{sa}(1+\mu) \gtreqless \frac{2\sigma_s}{\sigma_a^2 r^*} + \frac{r_{sa}}{p}$$

Case 1. If $r_{sa} = 0$, $dp/d\sigma_s < 0$

Case 2. If $r_{sa} > 0$, divide by r_{sa} and rearrange to

$$\frac{dp}{d\sigma_s} \gtreqless 0 \quad \text{as} \quad \mu - \frac{2\sigma_s}{\sigma_a r_{sa}} \gtreqless \frac{1}{p} - 1$$

Note: $p \gtreqless 1$ as $\mu \gtreqless \sigma_s/\sigma_a r_{sa}$
If $p = 1$, $dp/d\sigma_s < 0$.
If $0 < p < 1$, $\mu - \sigma_s/\sigma r_{sa} < 0$ and $1/p - 1 > 0$, implying $dp/dr_{sa} < 0$.

Case 3. If $r_{sa} < 0$.

$$\frac{dp}{d\sigma_s} \gtreqless 0 \quad \text{as} \quad p\left[r_{sa}(1+\mu) - \frac{2\sigma_s}{\sigma_a}\right] \gtreqless r_{sa}$$

Substitute for p and simplify. Since the right side is positive, if $r_{sa} < 0$, $dp/d\sigma_s < 0$.

$$\frac{dp}{d\sigma_s} \gtreqless 0 \quad \text{as} \quad r_{sa}\left[\mu^2 + \frac{\sigma_s^2}{\sigma_a^2 r^*}\right] \gtreqless \frac{2\mu\sigma_s}{\sigma_a}$$

$$\frac{dp}{d\sigma_s} \gtreqless 0 \quad \text{as} \quad r_{sa} \gtreqless \frac{\frac{2\mu\sigma_s}{\sigma_a}}{\mu^2 + \frac{\sigma_s^2}{\sigma_a^2 r^*}}$$

III. Proof that $dp/dr_{sa} \gtreqless 0$ as $\mu \gtreqless \sigma_s/\sigma_a r^*$

$$\frac{dp}{dr_{sa}} = \frac{\frac{-\sigma_s}{\sigma_a r^*}}{DEN} - \frac{NUM}{DEN^2}\left[\frac{-\sigma_s(1+\mu)}{\sigma_a r^*}\right]$$

$$\frac{dp}{dr_{sa}} \gtreqless 0 \quad \text{as} \quad \frac{1}{DEN}\left[-\frac{\sigma_s}{\sigma_a r^*} + \frac{p\sigma_s(1+\mu)}{\sigma_a r^*}\right] \gtreqless 0$$

Assuming $DEN > 0$,

$$\frac{dp}{dr_{sa}} \gtreqless 0 \quad \text{as} \quad p \gtreqless \frac{1}{1+\mu}$$

Substitute the definition of p, cross-multiply and simplify to

$$\frac{dp}{dr_{sa}} \gtreqless 0 \quad \text{as} \quad \mu^2 \gtreqless \frac{\sigma_s^2}{\sigma_a^2 r^*}$$

$$\frac{dp}{dr_{sa}} \gtreqless 0 \quad \text{as} \quad \mu \gtreqless \frac{\sigma_s}{\sigma_a r^*}$$

This condition is the same as

$$\mu \gtreqless \sigma$$

$$\mu \gtreqless \sigma_s/\sigma_a$$

$$\text{or } m_s \gtreqless m_a/\sqrt{r^*}$$

If p is relatively large (small), $dp/dr_{sa} > 0$ ($dp/dr_{sa} < 0$).

Appendix E: Transactions Costs and a Superior Asset

With a superior asset and transactions costs proportion to α, the investor desires to maximize m_s, where

$$m_s = \frac{\mu_p - R_f - C(n_s + 1)}{\sigma_p}$$

n_s = number of average assets in a portfolio including a superior asset and the total cost per dollar of wealth $= C(n+1) = D(n_s + 1)/W$. The portfolio mean is μ_p and the portfolio variance is σ_{ps}^2 where

$$\mu_p = p\mu_s + (1-p)\mu_a$$

$$\sigma_{ps}^2 = p^2\sigma_s^2 + (1-p)^2\sigma_a^2\left[r + \frac{1-r}{n_s}\right] + 2p(1-p)\sigma_s\sigma_a r_{sa}$$

The first order conditions are to take dm_s/dp and dm_s/dn_s and set equal to zero. In general, the results are very complex because the percentage cost $C(n_s+1)$ depends upon the proportion invested in the superior asset.

However, for two cases the percentage cost does not change as the proportion invested in the superior asset varies. (1) For large volume discount brokers, the brokerage costs are simply a percent of total wealth. (2) For simple linear cost functions, the percent brokerage cost can be shown to be independent of the proportion invested in the superior asset. In these cases, the results for dm_s/dp are very similar to Appendix C. We find that

$$p = \frac{\mu^* - \sigma r_{sa}}{\mu^* + \sigma^2 - r_{sa}\sigma(1+\mu^*)}$$

where

$$\mu^* = \frac{\mu_s - R_f - C(n_s + 1)}{\mu_a - R_f - C(n_s + 1)}$$

For dm_s/dn_s

$$\frac{dm_s}{dn_s} = \frac{C'}{\sigma_p} + \frac{[\mu_p - R_f - C(n_s + 1)](1 - p)^2 \sigma_a^2 (1 - r)}{2\sigma_p^3 n_s^2}$$

where $C' = dC(n_s + 1)/dn_s$. Set dm/dn_s equal to zero and solve.

$$n_s^2 = \left[\frac{.5\sigma_a^2(1 - r)}{C'}\right]\left[\frac{m_s}{\sigma_{ps}}\right](1 - p)^2$$

Without a superior asset, $p = 0$ and

$$n^2 = \left[\frac{.5\sigma_a^2(1 - r)}{C'}\right]\left[\frac{m}{\sigma_p}\right]$$

Comparing these two equations, the impact of a superior asset depends upon two effects.

i. Wealth effect: $(1 - p)^2$. Since $(1 - p)^2$ is smaller than 1, this effect tends to reduce n_s compared to n.
ii. Diversification effect: Since we expect that $m_s/\sigma_{ps} > m/\sigma_p$, this effect tends to increase n_s.

The net impact of the introduction of a superior asset upon the number of average assets depends upon both effects. An extensive numerical evaluation has been found that the wealth effect dominates for $r_{sa} \geq 0$. That is, a superior asset reduces the optimal number of average assets as long as $r_{sa} \geq 0$.

XI. NOTES ON CONTRIBUTORS/ACKNOWLEDGMENTS

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