The Effect of Coupon Level on Treasury Bond Futures Delivery*

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A considerable body of literature has developed concerning the cheapest bond to deliver against the Chicago Board of Trade Treasury Bond futures contracts. The investor who is short in this contract has the option to deliver one out of many possible bonds. A number of authors have argued that this so-called quality option will affect the futures price before delivery.[1] A second issue has concerned the impact of maturity and coupon level upon cheapness of delivery, (Jones, 1985, Kane and Marcus, 1984, Kilcollin, 1982, Kolb, Gay and Jordan, 1982, Livingston, 1984, and Meisner and Labuszewski, 1984) that is, are high or low coupon, long or short maturity, cheapest to deliver.

This article will focus on the impact of coupon level upon cheapness in delivery. If all bonds of a particular maturity have the same yield to maturity, it has been shown that low (high) coupon bonds would be better to deliver if yield to maturity is higher (less) than 8%. If bonds of the same maturity have different yields to maturity because of term structure and/or tax effects, the analysis is considerably more complex and the literature is unclear about the cheapest to deliver.[2]

This article derives the impact of bond coupon level upon cheapest to deliver analysis for the cases where higher (lower) coupon bonds have higher yields to maturity, called a positive (negative) coupon effect. It will be shown that there is a critical point which is above (below) 8% for positive (negative) coupon effects. If bond yields exceed this critical point, the lowest coupon bonds will be cheapest to deliver. If yields exactly equal this critical point, all bonds will be equivalent in delivery value. If yields are less than the critical point (but still possibly above 8%), the highest coupon bonds are cheapest to deliver.

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[1]See Garbade and Silber (1983), Gay and Manaster (1984) and Kane and Marcus (1987). Livingston (1987) argues that the so-called quality option would have a zero value in perfect markets.

[2]Kilcollin (1982) and Livingston (1984) suggest that high coupon bonds are best to deliver for positive coupon effects.

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These results have the interesting implication that the level of interest rates will significantly alter value in delivery. If interest rates are considerably above 8% (e.g., 12%), low coupon bonds will be cheapest to deliver. For positive coupon effects, if rates of interest drop below the critical point but still remain above 8%, the highest coupon bonds will be cheapest to deliver.

This article considers the impact of coupon level upon cheapest bond to deliver for various term structures, holding maturity constant. Previous research (Kilcollin, 1982 and Livingston, 1984) has considered the impact of maturity upon cheapest deliverable, holding coupon constant. In practice, the cheapest to deliver depends upon coupon level and maturity. In addition, the availability of bonds of various coupons levels and maturities will significantly affect the cheapest bond to deliver. For example, it might be that, holding maturity constant, the lowest coupon bonds would be cheapest to deliver; and, holding coupon constant, the longest maturity bonds would be cheapest to deliver. The overall cheapest to deliver will not necessarily be the lowest coupon bond or the longest maturity. With restricted supply of deliverable bonds, the cheapest deliverable bond depends upon all three factors: the coupon level, maturity and price of the available deliverable bonds.

I. ASSUMPTIONS AND BACKGROUND

We will make the following assumptions: default-free bonds, noncallable bonds, $1 par values, no transactions costs, one delivery date[3] for futures and an instantaneous delivery process.[4]

For the CBT Treasury Bond futures contract, U.S. Treasury Bonds with at least 15 years to maturity or to first call date can be delivered. The amount paid for delivering a particular bond is called the invoice price and is equal to the settlement futures price (denoted by F) times a conversion or adjustment factor (denoted by Q). For a bond with maturity of n periods, coupon rate of c (i.e., actual coupon divided by par),[5]

$$Q = \sum_{j=1}^{n} \frac{c}{(1.08)^j} + \frac{1}{(1.08)^n} \tag{1}$$

The proceeds to the short of delivering a particular bond with price B and $1 par value will be[6]

$$\text{Proceeds} = FQ - B \tag{2}$$

[3]In practice, there is a delivery month.

[4]In practice, the delivery process involves three days and the wildcard and end-of-month options. (Kane and Marcus, 1986).

[5]In practice, semiannual coupons are used, n is rounded down to the nearest number of quarters from the first day of the delivery month to the smaller of maturity or first call date, and the factor is rounded to four decimal places.

[6]This overlooks accrued interest. However, the results are identical with accrued interest. The long must pay the accrued interest. Either the short buys the bond and pays the purchase price plus accrued interest or the short owns the bond and is entitled to the accrued interest. If accrued interest is denoted by I, then Equation (2) becomes:

$$\text{Proceeds} = F \cdot Q + I - (B + I) = FQ - B$$

The accrued interest terms cancel.

II. THE CHEAPEST TO DELIVER

A bond with higher proceeds would be better for the short to deliver. In equilibrium in a perfect market, the proceeds would be zero for the cheapest bond to deliver since positive proceeds for any bond would result in immediate arbitrage profits from shorting futures and delivering that bond.

To see the impact of coupon upon relative delivery value, take the derivative of delivery proceeds with respect to coupon.

$$\frac{d(\text{proceeds})}{dc} = F\frac{dQ}{dc} - \frac{dB}{dc} \tag{3}$$

To evaluate this expression, note that the price of a bond with $1 par value can be written as

$$B = cA + D \tag{4}$$

where A represents the present value of an n period annuity of $1 per period and D is the present value of $1 received in n periods. A and D incorporate term structure and tax effects.

From Equation (4) the derivative of B with respect to c equals A. From Equation (1), dQ/dc equals $\Sigma_{j=1}^{n} (1.08)^{-j}$.

In a perfect market, delivery proceeds will equal zero for the cheapest bond to deliver, implying that the futures price F equals the ratio B^*/Q^* for the cheapest bond to deliver. The bond that makes the ratio of bond price divided by adjustment factor a minimum is the cheapest to deliver.[7] Therefore,

$$F = \frac{B^*}{Q^*} = \frac{c^*A + D}{c^* \sum_{j=1}^{n} (1.08)^{-j} + (1.08)^{-n}} \tag{5}$$

Substituting into Equation (3) for F, dQ/dc, and dB/dc results in

$$\frac{d(\text{proceeds})}{dc} = \frac{[c^*A + D] \sum_{j=1}^{n} (1.08)^{-j}}{c^* \sum_{j=1}^{n} (1.08)^{-j} + (1.08)^{-n}} - A \tag{6}$$

Algebraic manipulation implies that

$$\frac{d(\text{proceeds})}{dc} \gtreqless 0 \text{ as } (1.08)^{n} \sum_{j=1}^{n} (1.08)^{-j} \gtreqless \frac{A}{D} \tag{7}$$

Write A and D in terms of discount rates.

$$A = \sum_{j=1}^{n} \frac{1}{(1 + a)^{j}} \tag{8}$$

$$D = \frac{1}{(1 + d)^{n}} \tag{9}$$

[7]See Livingston (1984) for a proof.

Then, substitute for A and D in Equation (7), resulting in

$$\frac{d(\text{proceeds})}{dc} \gtreqless 0 \text{ as } \frac{(1.08)^n - 1}{.08} \gtreqless \frac{(1 + d)^n[1 - (1 + a)^{-n}]}{a} \tag{10}$$

The left side of Equation (10) is the future value of an annuity for n periods at the discount rate of 8%. The right side of Equation (10) is the future value of an annuity at the combined rates of d and a, the rates on a zero coupon bond and a pure annuity. It is convenient to write this latter annuity in terms of the discount rate q such that;

$$\frac{(1 + q)^n - 1}{q} = \frac{(1 + d)^n[1 - (1 + d + \Delta)^{-n}]}{d + \Delta} \tag{11}$$

where $a = d + \Delta$. Substituting into Equation (10), we see that,

$$\frac{d(\text{proceeds})}{dc} \gtreqless 0 \text{ as } .08 \gtreqless q \tag{12}$$

Neutral Coupon Effect

If all bonds with the same maturity have the same yield to maturity, the effect of coupon upon yield to maturity can be described as neutral. In this case, Δ is zero and $q = d = a$. If bond yields are greater (less) than 8%, d (proceeds)/dc is negative (positive). Since it is cheaper to deliver bonds with higher proceeds, low (high) coupon bonds are cheaper to deliver if yields are above (below) 8%.

Positive Coupon Effect

If higher coupon bonds have higher yields to maturity, we will describe the coupon effect as positive. In this case, Δ is positive and $d < a$; that is, the yield on a zero coupon bond d must be less than the yield on a pure annuity a. Since the yield y on a coupon-bearing bond must be between d and a, it must be that $d < y < a$. Empirically, positive coupon effects seem to be quite common. A positive coupon effect is illustrated by Table I.

The Appendix shows for positive coupon effects that $q < d < y < a$; that is, q is less than the yields on all bonds. For positive coupon effects it is possible that

Table I
COUPON VERSUS YIELD FOR A POSITIVE COUPON EFFECT ASSUMING MATURITY = 15, d = .06, a = .08

Coupon	Yield to Maturity
.00	.0600
.02	.0637
.04	.0657
.06	.0675
.08	.0689
.10	.0700

q is less than 8% even though yield to maturity exceeds 8%. Therefore, yields above 8% will not guarantee that low coupon bonds will be cheapest to deliver. This is illustrated below.

Figure 1 presents three graphical examples. In each example, the difference between a and d (Δ) is 2%. The horizontal axis shows coupons and yields to maturity. The vertical axis shows the ratio of bond price divided by adjustment factor (B/Q); recall that the smallest ratio is the cheapest bond to deliver.

In Figure 1, for the example on the extreme right q exceeds 8% and the lowest coupon bond is cheapest to deliver, since it has the smallest ratio of bond price to adjustment factor. In the middle example, q equals 8% and all bonds will have equivalent value in delivery since the ratio of bond price to adjustment factor is the same for all bonds. In the example on the left, yields are relatively low so that q is less than 8% and the highest coupon bonds are cheapest to deliver since they have the smallest ratio of B/Q.

From these examples, it is clear that yields considerably above 8% mean that low coupon bonds will be cheapest to deliver. As yields come down, at some point all bonds will be of equivalent delivery value (i.e., $q = 8\%$). For even lower yields, the highest coupon bond will be cheapest to deliver.

To provide insight into the levels of yields for which high or low coupon bonds are cheapest to deliver, Table II presents the values of zero coupon yields d for which q is 8% for various maturities and values of Δ ($\Delta = a - d$). For example, if $n = 15$ periods and Δ equals ½% (i.e., $a = d + .005$), then lower coupon bonds are cheapest to deliver if d is greater than .08383; all bonds are of equivalent delivery value is d equals .08383; and the highest coupon bonds are cheapest to deliver if d is less than .08383.

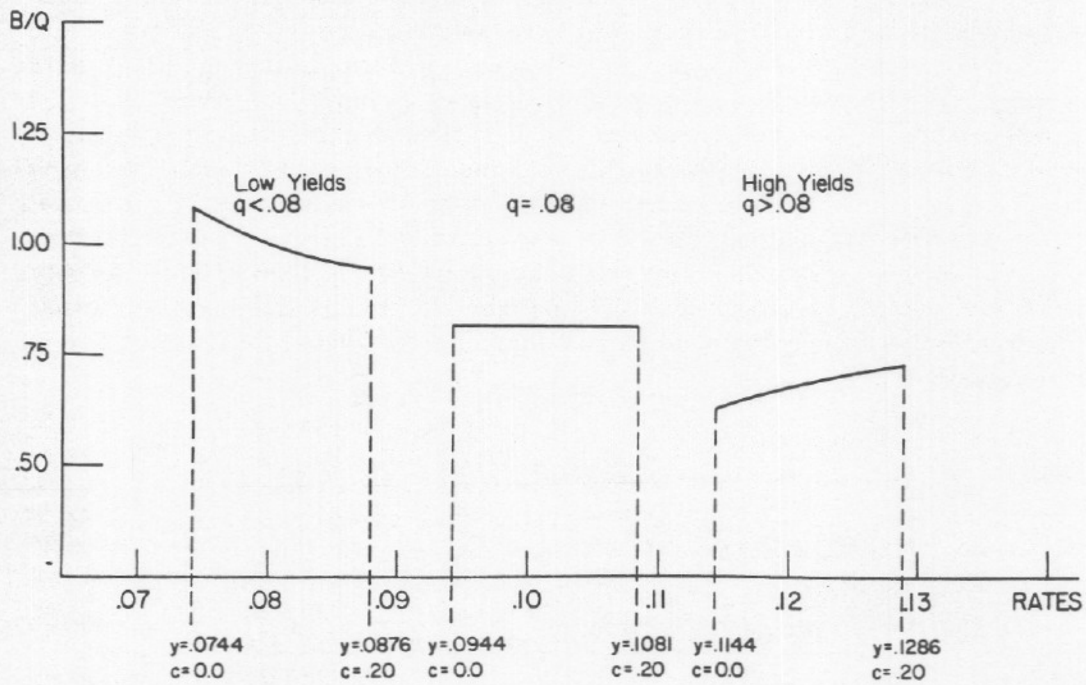

Figure 1
Positive Coupon Effect, 15 Years, $\Delta = +.02$

Table II
VALUES OF d FOR WHICH q EQUALS 8% FOR POSITIVE COUPON EFFECTS ($a > d$)

	Maturity (Years)			
$\Delta = a - d$	15	20	25	30
.005	.08383	.08327	.08284	.08249
.010	.08749	.08637	.08551	.08482
.015	.09100	.08931	.08803	.08701
.020	.09436	.09212	.09043	.08908

Negative Coupon Effects

If higher coupon bonds have lower yields, the coupon effect is negative. Then, the yield on a zero coupon bond d must be larger than the yield on an annuity a, and the yield on a coupon-bearing bond y must lie between d and a. That is, $d > y > a$. The Appendix shows that q will exceed d; that is, $q > d > y > a$.

If yields are high enough so that q exceeds 8%, then the lowest coupon bond (which has a high yield since $d > a$) has the smallest ratio of B/Q and is cheapest to deliver. If yields are such that q equals 8%, all bonds will have the same B/Q ratios and will be of equivalent value in delivery. If yields are quite low, the highest coupon bonds will have the smallest ratio of B/Q and will be cheapest to deliver.

III. CONCLUSION AND PRACTICAL CONSIDERATIONS

The previous section theoretically examined the impact of bond coupon level upon cheapness in delivery, holding maturity constant. Cheapness in delivery is also affected by maturity. Several papers (Kilcollin, 1982 and Livingston, 1984) have examined the impact of maturity upon cheapness, holding coupon constant.

In practice, cheapness in delivery depends upon the availability of bonds of various coupon levels and maturities. For example, it may be that the lowest coupon is cheapest to deliver, holding maturity constant, and the longest maturity is cheapest holding coupon constant. If the longest maturity has the lowest coupon, this bond will be cheapest to deliver. But if a shorter maturity bond has a lower coupon, this bond may be cheaper to deliver. Consequently, the availability of particular coupon levels and maturities will have a large impact upon cheapness in delivery in practice.

Appendix

Proof that $dq/d\Delta < 0$:
Using the definitions of q and Δ, implicit differentiation results in

$$\frac{dq}{d\Delta} = \frac{\dfrac{n(1 + d)^n(1 + d + \Delta)^{-(n+1)}}{d + \Delta} - \dfrac{(1 + d)^n[1 - (1 + d + \Delta)^{-n}]}{(d + \Delta)^2}}{\dfrac{n(1 + q)^{n-1}}{q} - \dfrac{[(1 - q)^n - 1]}{q^2}}$$

The denominator is positive if

$$n > \frac{(1 + q)[1 - (1 + q)^{-n}]}{q}$$

$$n > (1 + q)\left[\frac{1}{(1 + q)} + \cdots + \frac{1}{(1 + q)^n}\right]$$

$$n > 1 + \frac{1}{(1 + q)^1} + \cdots + \frac{1}{(1 + q)^{n-1}}$$

This will be true for all positive values of q. Consequently, the sign of the derivative is the sign of the numerator. The numerator will be negative if

$$\frac{n(1 + d)^n(1 + d + \Delta)^{-(n+1)}}{d + \Delta} < \frac{(1 + d)^n[1 - (1 + d + \Delta)^{-n}]}{(d + \Delta)^2}$$

Simplifying

$$n(1 + d + \Delta)^{-(n+1)} < \frac{1 - (1 + d + \Delta)^{-n}}{d + \Delta}$$

The right side is the present value of an n period annuity discounted at the rate $d + \Delta$. The left side is n times the present value of an n period zero coupon bond discounted at the rate $d + \Delta$. Clearly the left side is less than the right side.

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