Relative Impact of Duration and Convexity on Bond Price Changes

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■ Duration is a first order approximation of the magnitude of a percentage change in a bond's price when interest rates change, and convexity can be employed to improve the approximation to second order. Duration and convexity are employed in a wide variety of applications from immunizing future liabilities to hedging a mortgage pipeline at a financial institution. Duration and convexity are important tools to assess and manage interest rate risk exposure. The percentage change in a bond's price with respect to a change in interest rates can be expressed via a Taylor series expansion (see section I below). An extensive literature has examined the first derivative term in the Taylor expansion, namely modified Macaulay's duration, as a measure of bond price volatility.[1] Some researchers have begun to examine the impact of the second derivative term, namely convexity, upon price risk.[2] The purpose of this paper is to examine the relative importance of duration and convexity in approximating bond price changes. Specifically, we identify when it is particularly important to examine convexity. We find that the relative importance of convexity rises with a decline in interest rates.

I. Taylor Series Expansion

For a flat term structure, the price of a bond P, with annual coupon C, par value F, maturity of n, and yield to maturity y is equation (1).

$$P = \sum_{j=1}^{n} \frac{C}{(1+y)^{j}} + \frac{F}{(1+y)^{n}} \tag{1}$$

The change in price for a change in yield may be expressed in terms of a Taylor Series expansion as follows:[3]

$$\Delta P = \sum_{n=1}^{m} \frac{P^{(n)}(y)}{n\,!} (\Delta y)^{n} + r_m(y),$$

where $r_m(y)$ is the remainder using the first m terms of the approximation. Setting m=2, expanding the summation, dividing by the price (P) to get percentage changes, and omitting the remainder, the approximation is written as equation (2).

$$\frac{\Delta P}{P} \approx \frac{dP}{dy} \cdot \frac{(\Delta y)}{P} + \frac{d^2p}{dy^2} \cdot \frac{(\Delta y)^2}{(P)(2\,!)} \tag{2}$$

The negative of the first term (without Δy) is often called modified Macaulay's duration. Exhibit 1 shows the relationship between price and yield to maturity. Duration is a measure related to the tangent to this curve at a particular level of interest rates. Duration can be used as a measure of the sensitivity of bond price to changes in yield to maturity. Geometrically, this means moving along the tangent. For small changes in yield, moving along the tangent is fairly close to moving along the curve itself.

Adding higher derivative terms increases the accuracy of the approximation, as shown in Exhibit 1. For decreases in interest

1. See Bierwag [1,2], Bierwag and Kaufman [3], Boquist et. al. [4], Fabozzi and Fabozzi [7], Fisher and Weil [8], Grove [10], Hawawini [11], Hicks [12], Homer and Liebowitz [13], Hopewell and Kaufman [14], Livingston and Caks [16], Livingston [17], [18], and [19], Macaulay [20], Malkiel [21], Redington [23], and Samuelson [24].

2. See Dunetz and Mahoney [5], Grantier [9], and Nawalka and Lacey [22]. For an application of convexity to equities, see Johnson [15].

3. Let P=P (y). Then the Taylor Series may be expressed as

$$P(y') = \sum_{n=0}^{\infty} \frac{P^{(n)}(y)}{n\,!} (y' - y)^{n}$$

where $P^{(n)}(y)$ denotes the nth derivative. Let $\Delta y = y' - y$, and $\Delta P = P(y') - P(y)$, and rearranging we have

$$P(y') = P(y) + \sum_{n=1}^{\infty} \frac{P^{(n)}(y)}{n\,!} (\Delta y)^{n} \text{ and } \Delta P = \sum_{n=1}^{\infty} \frac{P^{(n)}(y)}{n\,!} (\Delta y)^{n}$$

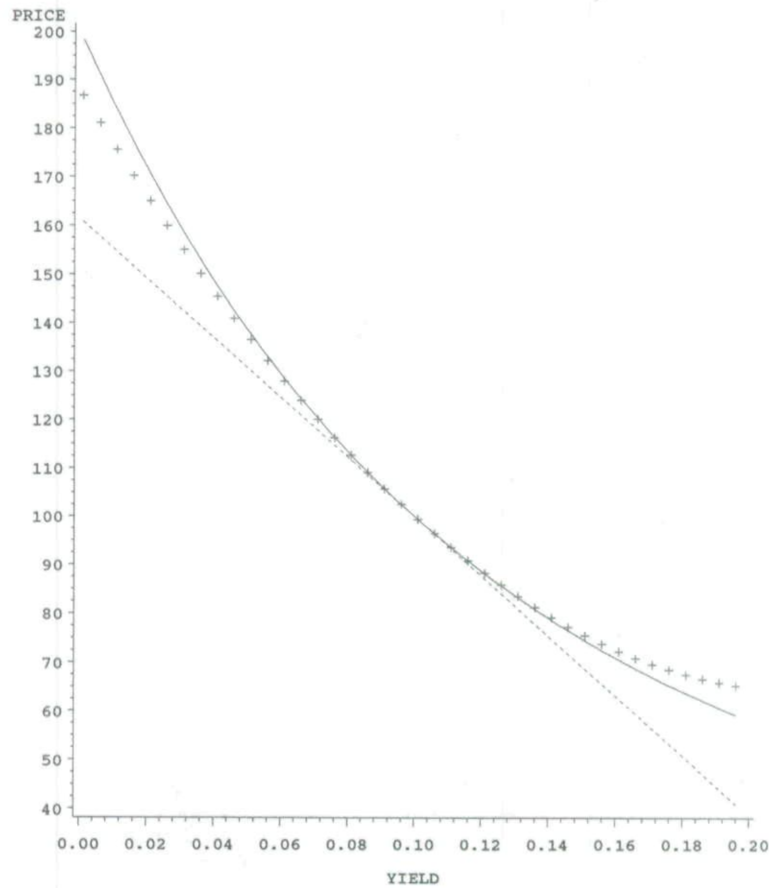
The Taylor Series may also be expressed as

$$\Delta P = \sum_{n=1}^{m} \frac{P^{(n)}(y)}{n\,!} (\Delta y)^{n} + r_m(y) \text{ where } r_m(y) = \frac{f^{(m+1)}(t_y)}{(m+1)\,!} (\Delta y)^{m+1} \text{ and}$$

$y < t_y < y'$ if $y' > y$
$y' < t_y < y$ if $y' < y$.

(For more information, see Ellis and Gulick [6] or any calculus book.)

Exhibit 1. Price-Yield Relation Based on Duration, Convexity and Actual



NOTE: - Actual, -- Duration, ++ Convexity

rates, the approximation gets closer to the actual price from below as terms are added. The reason is that odd numbered derivatives are negative. When interest rates decrease, Δ y to an odd power is negative, making the product positive for odd terms. All the even terms are positive. This implies that as terms are added, the total approximation increases. That is, the total approximation approaches the true value from below. For increases in interest rates, $(\Delta y)^i$ is positive for all powers of i. Since the odd numbered derivatives are positive and the even ones are negative, adding the odd terms reduces the sum and adding the even terms increases the sum. This means that the approximation oscillates around the true value as terms are added to the Taylor Series expansion.[4]

II. Duration and Convexity

Convexity sharpens the approximation of bond price changes since duration is only a first approximation. This paper will examine the relative value of adding the second term, or convexity, for a variety of bond types. The goal is to determine the extent of improvement in accuracy by adding convexity. Assume that yield to maturity changes by Δ y. Then, the percentage change in price can be written as equation (3).

$$\frac{\Delta P}{P} = \frac{\sum_{j=1}^{n} \frac{C}{(1+y+\Delta y)^j} + \frac{F}{(1+y+\Delta y)^n}}{\sum_{j=1}^{n} \frac{C}{(1+y)^j} + \frac{F}{(1+y)^n}} - 1 \qquad (3)$$

As shown in Appendix A, the first derivative (negative of modified duration) can be written as equation (4).

$$-D \equiv \frac{\frac{dP}{dy}}{P} = -\frac{\left\{\frac{C}{y^2}\left[1-\frac{1}{(1+y)^n}\right]+\frac{n(F-C/y)}{(1+y)^{n+1}}\right\}}{P} \qquad (4)$$

As shown in Appendix A, the second derivative (convexity) can be written as equation (5).

$$V \equiv \frac{\frac{d^2P}{dy^2}}{2P} =$$

$$\frac{\frac{2C}{y^3}\left[1-\frac{1}{(y^2(1+y)^n}\right]-\frac{2Cn}{n^2(1+y)^{n+1}}+\frac{n(n+1)(F-C/y)}{(1+n)^{n+2}}}{2P} \qquad (5)$$

4. It should be noted that the addition of higher order terms will always increase the precision of the approximation. Thus, although oscillation occurs when $\Delta y > 0$, the estimation is increasingly accurate.

Exhibit 2. Modified Duration and Convexity

Security Type	Modified Duration (D)	Convexity (V)
Zero Coupon Bond $C = 0$	$\frac{n}{1+y}$	$\frac{n(n+1)}{2(1+y)^2}$
Perpetual Bond $n = \infty$	$\frac{1}{y}$	$\frac{1}{y^2}$
Par Bond $P = F$	$\frac{1}{y}\left[1 - \frac{1}{(1+y)^n}\right]$	$\frac{1}{y^2}\left[1 - \frac{1}{(1+y)^n}\right] - \frac{n}{y(1+y)^{n+1}}$
Annuity Bond $F = 0$	$\frac{1}{y} - \frac{n}{(1+y)[(1+y)^n - 1]}$	$\frac{1}{y^2} - \frac{n}{(1+y)[(1+y)^n - 1]}\left[\frac{1}{y} + \frac{n+1}{2(1+y)}\right]$

The modified duration and convexity for zero coupon bonds, par bonds, perpetual bonds and annuities are shown in Exhibit 2.

III. The Relative Importance of Convexity

In this section we derive a measure of the relative importance of convexity and examine some of its properties. We can rewrite the Taylor Series expansion in equation (2) as equation (6).

$$\frac{\Delta P}{P} = -(D)\,\Delta y + (V)\,\Delta y^2 + r_2(y) \tag{6}$$

Dividing both sides by $\frac{\Delta P}{P}$, we have equation (7).

$$1 = \frac{-(D)\Delta y}{\Delta P/P} + \frac{(V)\,\Delta y^2}{\Delta P/P} + \frac{r_2(y)}{\Delta P/P} \tag{7}$$

The first term on the right hand side (RHS) is the proportion of the percentage price change explained by the modified duration component of the Taylor series. Similarly, the second term on the RHS of equation (7) is the proportion of the percentage price change explained by the convexity component of the Taylor series. In order to measure the relative importance of convexity and duration we examine the following ratio:

$$R \equiv \frac{(V)\,\Delta y^2/(\Delta P/P)}{-(D)\,\Delta y/(\Delta P/P)} = -\frac{V}{D}\Delta y\,. \tag{8}$$

Substituting equation (4) and (5) into (8) and rearranging (see Appendix B) we have equation (9).

$$R = -\Delta y \left\{ \frac{cd^{n+2} - cd^2 - nCyd + (1/2)\,n\,(n+1)\,(F - c/y)\,y^3}{yd\,[\,cd^{n+1} - cd + n\,(F - c/y)\,y^2\,]} \right\} \tag{9}$$

where

$d = (1 + y)$.

Exhibit 3 provides an explicit expression as well as comparative statics for the four special cases given in Exhibit 2. The following line of reasoning shows that the ratio of convexity to duration is close to zero for short maturity bonds and can be large for long maturity bonds. Short-term bonds are quite similar to zero coupon bonds. Thus, for short-term bonds, the ratio of convexity to duration is approximately - $[(n+1)\Delta y]/[2(1+y)]$. For example, for one-period bonds (i.e., n=1), the ratio is - $\Delta y/(1+y)$, which is relatively small. In general, the ratio is close to zero. Convexity is relatively small compared to duration.

Long-term bonds are quite similar to perpetual bonds. Thus, the ratio of convexity to duration for long-term bonds is approximately the same as the ratio for perpetual bonds, namely - $\Delta y/y$. The ratio for long-term bonds may be quite sizable, because convexity can be large relative to duration. Exhibit 4 presents an example of the ratio of convexity to duration as coupon and maturity vary.[5] This is exactly the result predicted by the ratios of convexity to duration for zero coupon and perpetual bonds in Exhibit 3. Short-term bonds behave like zeroes and long-term bonds like perpetuities.

Exhibit 5 illustrates the relative importance of convexity with respect to coupon and yield for 10-year bonds and a one percent decline in yield. We see that the relative importance of convexity declines with yield and increases slightly with coupon. Exhibit 6 presents the ratio of convexity to duration as Δy (DELTA) and maturity vary. As the absolute value of Δy increases, so does R. Exhibit 7 presents Δy and yield. R is very sensitive to Δy but not to Y.

IV. Conclusion

The relative impact of bond duration and convexity upon bond price changes is examined. The relative importance of convexity is shown to decrease as interest rates increase. Also, three dimensional graphs are generated to illustrate the relative importance of convexity over a range of different parameters. These results are limited, because we assume a flat term structure. A logical extension of our analysis is to cases where the term structure is not flat and changes in yield are not uniform across the term structure. ■

5. Exhibits 4 and following portray negative Δy (yield declines) because positive deltas oscillate, making the presentation much more complex. Also, the choice of 1% is arbitrary since the ratio is linear in Δy. (See Exhibit 3.)

Exhibit 3. A Measure of the Relative Importance of Convexity with Comparative Statics

Security Type ($\Delta y<0$)	R	Comparative Statics (dR/dy)
Zero Coupon Bond	$\frac{-(n+1)\Delta y}{2(1+y)} > 0$	$\frac{(n+1)\Delta y}{2(1+y)^2} < 0$
Perpetual Bond	$\frac{-\Delta y}{y} > 0$	$\frac{\Delta y}{y^2} < 0$
Par Bond	$\frac{-\Delta y}{y}\left[\frac{E}{D}\right] > 0$	$\left[\frac{\Delta y}{yD}\right]\left\{E\,[1/y + \frac{D'}{D}] - E'\right\} \gtrless 0$
Annuity	$\frac{-\Delta y}{y}\left[\frac{A}{B}\right] > 0$	$\left\{\frac{\Delta y}{yB}\right\}\left\{A\,[1/y + \frac{B'}{B}] - A'\right\} \gtrless 0$

$$A = (1+y)^{n+2} - (1+y)^2 - ny(1+y) - 1/2 \cdot n(n+1)y^2$$
$$B = (1+y)[(1+y)^{n+1} - (1+y) - ny]$$
$$E = (1+y)^{n+1} - (1+y) - ny$$
$$D = (1+y)^{n+1} - y - 1$$

$$E' = \frac{\partial E}{\partial y} = (n+1)(1+y)^n - n - 1$$
$$D' = \frac{\partial D}{\partial y} = (n+1)(1+y)^n - 1$$
$$A' = \partial A/\partial y = (n+2)(1+y)^{n+1} - 2(1+y) - n(1+2y) - n(n+1)y$$
$$B' = \frac{\partial B}{\partial y} = (n+2)(1+y)^{n+1} - 2(1+y) - n(1+2y)$$

Exhibit 4. Effect of Coupon and Maturity on the Ratio of Convexity to Duration. 10% Initial Yield, 1% Decline in Yield

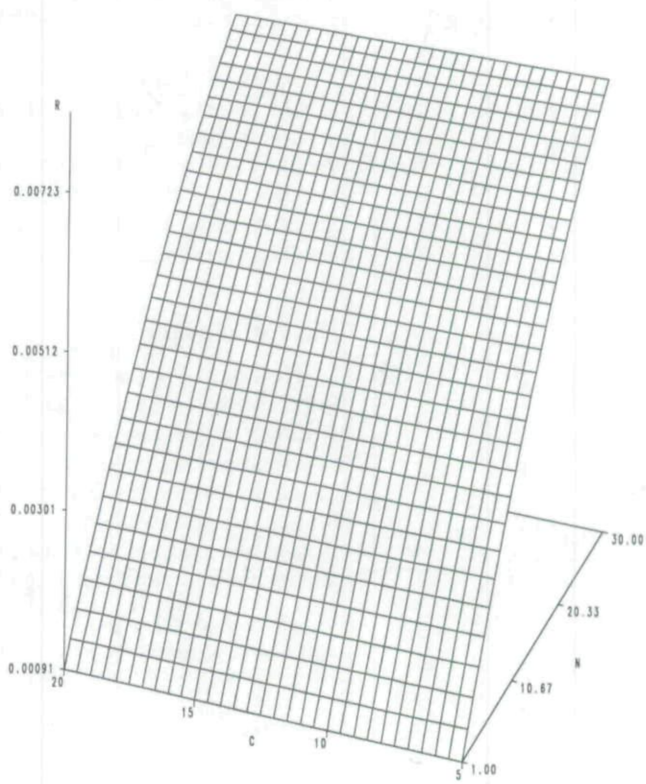


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Exhibit 5. Effect of Coupon and Yield on the Ratio of Convexity to Duration. 1% Decline in Yield, 10 Year Bonds

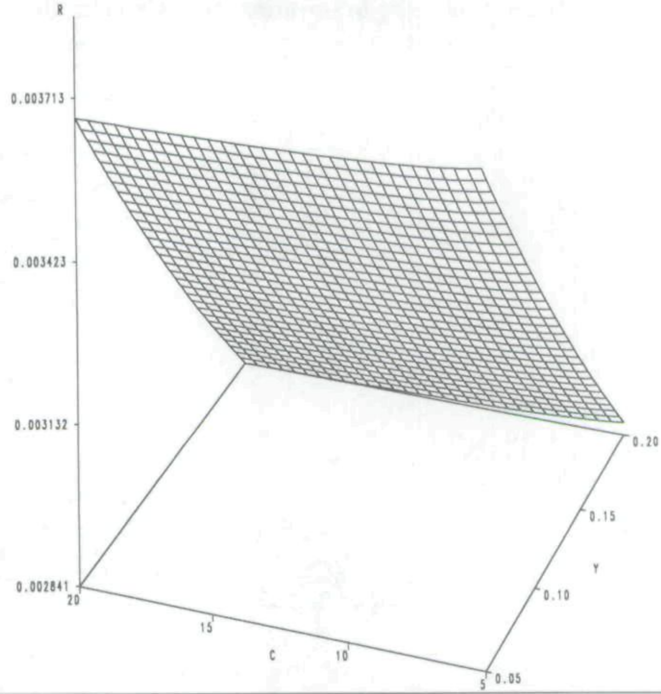


Exhibit 6. Effect of Delta and Maturity on the Ratio of Convexity to Duration. 10% Coupon, 10% Initial Yield

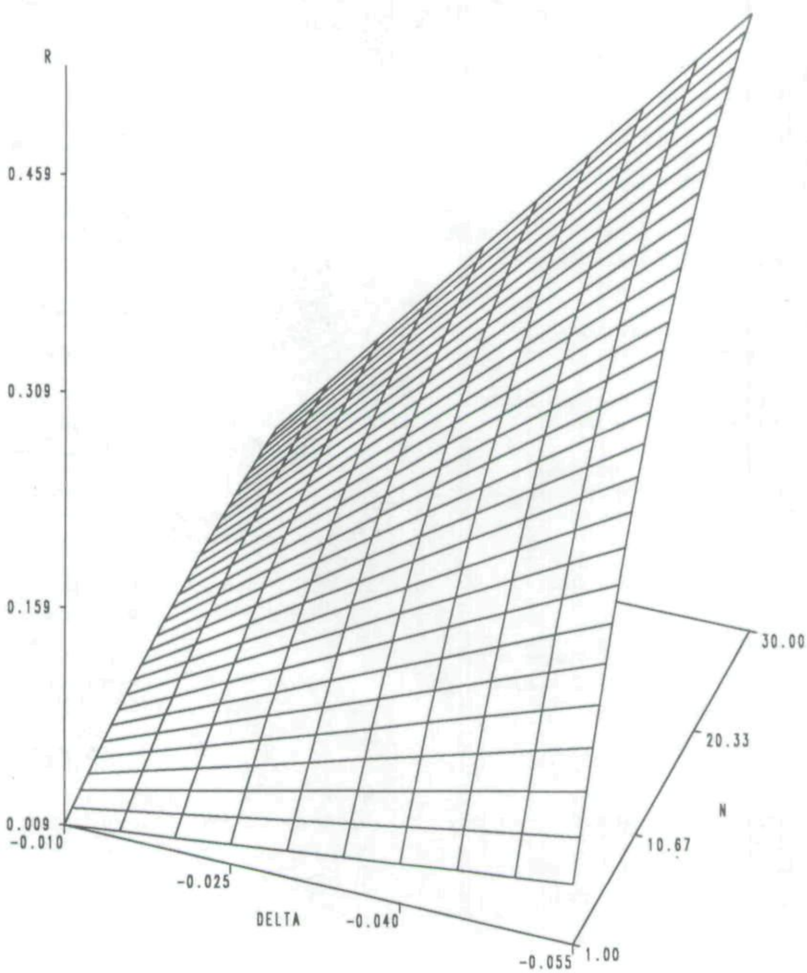
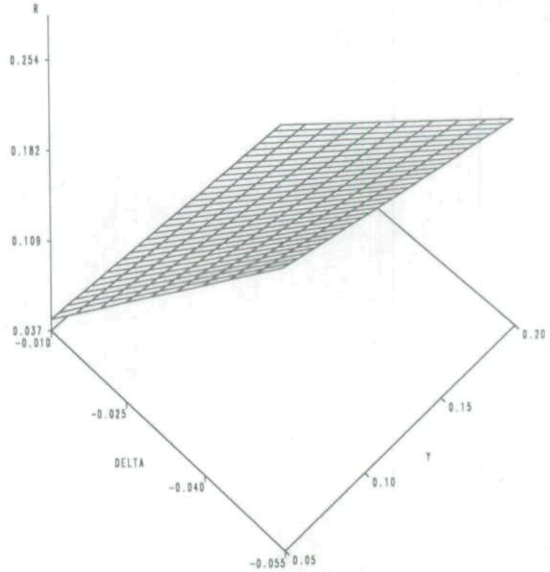


Exhibit 7. Effect of Delta and Yield on the Ratio of Convexity to Duration. 10% Coupon, 10 Year Maturity



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Appendix A

Using the Geometric Series Theorem, the price of a bond can be expressed as

$$P = \frac{C}{y}\left[1 - \frac{1}{(1+y)^n}\right] + \frac{F}{(1+y)^n}\,. \tag{A-1}$$

Thus, the first derivative is

$$\frac{dP}{dy} = -\frac{C}{y^2}\left[1 - \frac{1}{(1+y)^n}\right] + \frac{nC}{y\,(1+y)^{n+1}} - \frac{nF}{(1+y)^{n+1}} \tag{A-2}$$

$$= -\frac{C}{y^2}\left[1 - \frac{1}{(1+y)^n}\right] - \frac{n(F - C/y)}{(1+y)^{n+1}} \tag{A-3}$$

and the second derivative is

$$\frac{d^2P}{dy^2} = +\frac{2C}{y^3}\left[1 - \frac{1}{(1+y)^n}\right] - \frac{nC}{y^2\,(1+y)^{n+1}}$$

$$-\left[\frac{+\,nC/y^2\,(1+y)^{n+1} - n\,(F - C/y)\,(n+1)\,(1+y)^n}{(1+y)^{2\,(n+1)}}\right]$$

$$= +\frac{2C}{y^3}\left[1 - \frac{1}{(1+y)^n}\right] - \frac{2nC}{y^2(1+y)^{n+1}}$$

$$+\ \frac{n\,(n+1)\,(F - C/y)}{(1+y)^{n+2}}\,. \tag{A-4}$$

Appendix B

From equation (8) in the text we have

$$R = -\frac{V}{D}\,\Delta y = (-\Delta y)\quad\frac{d^2P/dy^2}{2\,(-\,dP/dy)} \tag{B-1}$$

Substituting the derivatives with $d \equiv (1 + y)$ results in

$$R = -\frac{\Delta y}{2}\left\{\frac{\dfrac{2Cd^{n+2} - 2Cd^2 - 2nCyd + n\,(n+1)\,(F - C/y)\,y^3}{y^3 d^{n+2}}}{\dfrac{Cd^{n+1} - Cd + n\,(F - C/y)\,y^2}{y^2\,d^{n+1}}}\right\} \tag{B-2}$$

By the rule of ratios (and cancelling 2) we have

$$R = -\Delta y\left\{\frac{Cd^{n+2} - Cd^2 - nCyd + (1/2)\,n\,(n+1)\,(F - C/y)\,y^3}{yd\,[\,C\,d^{n+1} - Cd + n\,(F - C/y)\,y^2\,]}\right\} \tag{B-3}$$