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Source: *The Journal of Finance*, Vol. 34, No. 1 (Mar., 1979), pp. 189-196
Published by: Wiley for the American Finance Association
Stable URL: https://www.jstor.org/stable/2327152
Accessed: 17-07-2019 21:27 UTC

THE JOURNAL OF FINANCE • VOL. XXXIV, NO. 1 • MARCH 1979

Bond Taxation and the Shape of the Yield-to-Maturity Curve

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I

EMPIRICAL STUDIES[1] HAVE FOUND a wide variety of shapes of bond yield to maturity curves (showing yield to maturity versus maturity) for coupon-bearing bonds. It is widely believed that zero coupon bond yield curves will have similar shapes. The relationship between yield curves for zero coupon bonds and coupon-bearing bonds is important because the majority of the theoretical work on bond pricing is developed for zero coupon bonds, but virtually all bonds (with maturities exceeding one year) are coupon-bearing bonds.

This paper will analyze the relationship between the yield curves for coupon-bearing bonds and zero coupon bonds in a world with differential taxation of coupons and capital gains. The relationship between coupon-bearing and zero coupon yield curves (and forward rates) is quite complex in the general case. To provide some insights into the relationships, this paper will examine two special cases: (1) level before-tax zero coupon yield curves, and (2) level after-tax zero coupon yield curves; it will be assumed in both cases that tax rates and bond coupon levels will be the same for all maturities.

When before-tax zero coupon rates are constant for all maturities, it will be shown that the yield curve for coupon-bearing bonds will rise with maturity. When; after-tax zero coupon rates are constant, the yield curve for non-par bonds will take on a wide variety of shapes and the yield curve for par bonds will be flat, consequently a shift in the level of after-tax zero coupon rates will alter the shape of the yield curve. Because many of the non-par coupon-bearing yield curves will slope upward as zero coupon rates change, there will be a tendency to observe upward sloping yield curves for coupon-bearing bonds, unless non-par bonds are continuously replaced by par bonds. Thus, taxation per se can create an upward bias in observed yield curves.

These results indicate that (except for the case where the yield curve for par bonds is flat) coupon-bearing bond yield curves cannot be used to make inferences about the shape of zero coupon yield curves even in the case where bonds of different maturity have the same coupon level and the same tax rates. In the more general case where actual coupons vary with maturity and tax rates differ by maturity, it will be shown that no inferences at all should be made. Consequently, existing empirical work testing theories of the behavior of zero coupon bonds, but employing coupon-bearing bond data, must be regarded with considerable reservation.

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The thoughtful comments of Michael Granito have substantially improved this paper.
[1] See Kessel [4], Cagan [1], McCullough [9].

II

In this section, some results required for subsequent sections are presented. The following assumptions will be made. (1) Bonds are non-callable and default-free. (2) There are no transactions costs, and securities are perfectly divisible. (3) At least two discount bonds, and one premium bond exist for each maturity. (4) Tax rates of TP for regular income and TG for capital gains are the same for all individuals and all maturities. (5) There is unrestricted short-selling, and short-sales are taxed symmetrically with long positions.

The price of a coupon-bearing bond (PB) may be expressed in terms of after-tax zero coupon rates as follows[2]:

$$PB = \frac{C\sum_{j=1}^{N}\frac{(1-TP)}{(1+TR_j)^j}}{1-\frac{TG}{(1+TR_N)^N}} + \frac{F(1-TG)}{(1+TR_N)^N - TG} \tag{1}$$

where C = bond coupon level, F = face (par) value, N = maturity, TR_j = the after tax discount rate for a j period zero coupon bond. This formula assumes that the difference between bond price and face value will be taxed as a capital gain (for bonds selling below par) or a capital loss (for premium bonds) at maturity. If the bondholder elects to amortize the premium over par an alternate bond pricing equation can be developed[3].

The yield to maturity is the rate y satisfying the following

$$PB = \sum_{j=1}^{N}\frac{C}{(1+y)^j} + \frac{F}{(1+y)^N} \tag{2}$$

In the case of a par bond, $PB = F$, and we have

$$y_N^* = \frac{C}{F} = \frac{\left[1-\frac{1}{(1+TR_N)^N}\right]}{\left[\sum_{j=1}^{N}\frac{(1-TP)}{(1+TR_j)^j}\right]} \tag{3}$$

where y_N^* is the yield to maturity for a par bond of maturity N.

[2] (1) has been derived by Livingston [5]. An alternate derivation is as follows. The price of a coupon bearing bond must be PB where

$$PB = \sum_{j=1}^{N}\frac{C(1-TP)}{(1+TR_j)^j} + \frac{F-(F-PB)(TG)}{(1+TR_N)^N}$$

Solving for PB results in (1).

[3] If the premium over par is amortized on a straight line basis as a deduction from regular income, we have

$$PB = \sum_{j=1}^{N}\frac{C(1-TP)}{(1+TR_j)^j} + \frac{F}{(1+TR_N)^N} + \sum_{j=1}^{N}\left[\frac{PB-F}{N}\right]\left[\frac{TP}{(1+TR_j)^j}\right]$$

Denote the before-tax yield to maturity for a zero coupon bond as BR_j. Then TR_j and BR_j are related as follows[4]:

$$(1 + TR_j)^j = (1 + BR_j)^j(1 - TG) + TG \tag{4}$$

Many authors compute forward rates f computed from coupon-bearing bonds[5] satisfying (5).

$$1 + f_j = \frac{(1 + y_j)^j}{(1 + y_{j-1})^{j-1}} \tag{5}$$

Solving for PB

$$PB = \frac{\sum_{j=1}^{N} \frac{C(1 - TP)}{(1 + TR_j)^j}}{1 - \frac{TP}{N} \sum_{j=1}^{N} \frac{1}{(1 + TR_j)^j}} + \frac{\frac{F}{(1 + TR_N)^N} - \frac{TP}{N} \sum_{j=1}^{N} \frac{F}{(1 + TR_j)^j}}{1 - \frac{TP}{N} \sum_{j=1}^{N} \frac{1}{(1 + TR_j)^j}}$$

See Livingston [6] for a more thorough treatment of premium bonds.

[4] If PD is the current price of a zero coupon bond with \$1 face,

$$PD = \frac{1 - (1 - PD)(TG)}{(1 + TR_N)^N}$$

Solving for PD,

$$PD = \frac{(1 - TG)}{(1 + TR_N)^N - TG}$$

PD can also be written in terms of before-tax discount rates as

$$PD = \frac{1}{(1 + BR_N)^N}$$

Setting $PD = PD$ in the last two equations results in (4).

[5] A considerable literature deals with zero coupon bonds and forward rates in a *non*-tax world. In a tax world in which zero coupon bonds involve a capital gains tax liability, Scott [12] has shown that there are three forward rates associated with zero coupon bonds.

$$1 + tf_j = \frac{[1 + TR_j]^j}{[1 + TR_{j-1}]^{j-1}}$$

$$1 + zf_j = \frac{[1 + BR_j]^j}{[1 + BR_{j-1}]^{j-1}}$$

$$bf_j = \frac{tf_j}{1 - TG}$$

tf_j is the after tax one period forward rate from time $j - 1$ to time j. zf_j would be the usual forward rate computed from zero coupon bonds. bf_j is the before-tax one-period forward rate, if tf_j is the after-tax forward rate. Scott [12] has shown that

$$zf_j = \frac{tf_j}{1 - \frac{TG}{(1 + TR_{j-1})^{j-1}}} = \frac{(bf_j)(1 - TG)}{1 - \frac{TG}{(1 + TR_{j-1})^{j-1}}}$$

Notice that, if $TG = 0$, $zf_j = tf_j = bf_j$. If $TG > 0$, $zf_j < bf_j$, and $zf_j > tf_j$.

III

The concern of this section will be the relationship between coupon-bearing bond yield curves and before-tax zero coupon yield curves. In the case of constant before-tax zero coupon discount rates, we will prove that the yield curve for par bonds will slope upward. To see when the yield curve for par bonds has an upward slope, we must ask when y^*_{N+1} is greater than y^*_N, where y^*_{N+1} is the yield to maturity for an $N+1$ period par bond. That is, we must ask when the following inequality will hold.

$$\left[\frac{1-\dfrac{1}{(1+TR_{N+1})^{N+1}}}{\sum_{j=1}^{N+1}\dfrac{(1-TP)}{(1+TR_j)^j}}\right] > \left[\frac{1-\dfrac{1}{(1+TR_N)^N}}{\sum_{j=1}^{N}\dfrac{(1-TP)}{(1+TR_j)^j}}\right] \tag{6}$$

(6) will hold when[6]

$$(1+TR_{N+1})^{N+1} > (y_N)(1-TP)(1+TR_N)^N + (1+TR_N)^N \tag{7}$$

The slope of the yield curve for par bonds will be positive when (7) holds. (7) can be shown to hold whenever $TR_{N+1} > TR_N$ by the following line of reasoning. If $TR_{N+1} > TR_N$, then $(1+TR_{N+1})^N > (1+TR_N)^N$ and

$$(1+TR_{N+1})^{N+1} > (1+TR_N)^N + (1+TR_N)^N(TR_{N+1}) \tag{8}$$

Comparing (7) and (8), we see that (7) will hold as long as

$$(y_N)(1-TP)(1+TR_N)^N < (TR_{N+1})(1+TR_N)^N \tag{9}$$

Rewriting,

$$(y_N)(1-TP) < TR_{N+1} \tag{10}$$

If $TR_{N+1} > TR_N$, (10) will hold whenever

$$y_N < \frac{TR_N}{(1-TP)} \tag{11}$$

Substituting for y_N and rewriting,

$$\frac{1}{TR_N}\left[1-\frac{1}{(1+TR_N)^N}\right] < \sum_{j=1}^{N}\frac{1}{(1+TR_j)^j} \tag{12}$$

If the TR_j are monotonically increasing, (12) must hold because the lefthand side of (12) is the present value of an annuity discounted at the rate TR_N (which is the largest of the TR_j) and the righthand side is the present value of a stream discounted at the rates TR_j, less than or equal to TR_N.

Thus we have proved that the yield curve for par bonds will slope upward

[6] To go from (6) to (7) we can cross-multiply, cancel $(1-TP)$ on both sides, and rearrange to

$$\left[\frac{(1+TR_{N+1})^{N+1}-(1+TR_N)^N}{(1+TR_N)^N}\right]\left[\sum_{j=1}^{N}\frac{1}{(1+TR_j)^j}\right] > \frac{(1+TR_N)^N-1}{(1+TR_N)^N}$$

Dividing both sides by the summation term, using the definition of y^*_N from (3), and rearranging results in (7).

whenever the TR_j increase monotonically as j increases. It is proved below that the TR_j will be monotonically increasing whenever the BR_j are constant for all maturities. Consequently, the yield to maturity curve for par bonds will have a positive slope for constant before-tax zero coupon discount rates.

PROPOSITION: *If* $BR_1 = BR_2 = \cdots = BR_N = BR$, $TR_{j+a} > TR_j$ *for* $j \geq 1$, $a > 0$. *Using* (4), *the proposition will hold if*

$$[(1 + BR)^{j+a}(1 - TG) + TG]^{\left(\frac{j}{j+a}\right)} > (1 + BR)^j(1 - TG) + TG \tag{13}$$

At $BR = 0$, (13) *will hold as an equality. Therefore, the proposition will be proved if the derivative of the lefthand side of* (13) *is always greater than the derivative of the righthand side, i.e. if*

$$[(1 + BR)^{j+a}(1 - TG) + TG]^{\left(\frac{j}{j+a}-1\right)}[1 + BR]^{j+a-1} > [1 + BR]^{j-1} \tag{14}$$

This simplifies to

$$1 + TR_{j+a} > \frac{1}{(1 + BR)} \tag{15}$$

which must always hold for $BR > 0$ *and* $TR_{j+a} > 0$.

In the case of non-par bonds, the author has conducted an extensive numerical evaluation of alternative tax rates and zero coupon discount rates (BR_j). Except for bonds selling at enormous premiums, the yield curve was upward sloping[7]. Table 1 contains a typical case. The reader can easily verify that forward rates computed from the yields of coupon-bearing bonds in Table 1 will differ considerably from after-tax zero coupon forward rates (See footnote #5 above).

IV

Consider now the case of constant after-tax zero coupon rates. For par bonds, the yield curve will be flat and $y^* = TR/(1 - TP)$. But the yield curve for non-par bonds will not be flat. An extensive numerical evaluation was conducted[8]. A typical example is shown in Table 2 in which the after-tax zero coupon rate is 4%.

Quite remarkably, with constant after-tax zero coupon rates, non-par yield curves can be dish-shaped (i.e. declining and then rising), monotonically increasing or monotonically declining; the forward rates will have similar shapes. This means that upward or downward shifts over time in the (level) after-tax zero coupon curve will alter the shape of the existing coupon-bearing yield curve. For example, if all bonds were initially par bonds and the yield curve were flat, a rise in the (flat) after-tax zero coupon curve would result in a rising yield curve for the coupon-bearing bonds (now selling below par). A further rise in the after-tax

[7] Assuming bonds are priced by equation (1), the yield to maturity was computed for 1 through 20 years (assuming annual compounding) for tax rates of $TP = 2TG$. The BR_j were varied by 1% increments from 2% to 8%; coupons were varied by 1% increments from 2% to 8%; tax rates were varied by 10% increments from $TP = .10$ to $TP = .50$.

[8] The same assumptions as in Footnote #7 were made, except that the after-tax rate was varied by 1% increments from 2% to 8%.

Table 1
Bond Yield Percentage
Constant Before-Tax Zero Coupon Rate of 4%; TP = 30%; TG = 15%

Maturity	Coupon (%) 2	3	4	5	6	7	8	Coupon of Par Bond	After-Tax Zero Coupon Rate
1	4.3610	4.5372	4.7106	4.8813	5.0493	5.2146	5.3774	4.8571	3.3999
2	4.3701	4.5484	4.7225	4.8926	5.0586	5.2209	5.3794	4.8709	3.4098
3	4.3791	4.5596	4.7344	4.9037	5.0678	5.2269	5.3813	4.8844	3.4195
4	4.3881	4.5706	4.7459	4.9144	5.0765	5.2326	5.3829	4.8974	3.4290
5	4.3971	4.5814	4.7511	4.9247	5.0848	5.2377	5.3840	4.9100	3.4383
6	4.4061	4.5923	4.7683	4.9349	5.0928	5.2427	5.3851	4.9223	3.4474
7	4.4152	4.6030	4.7792	4.9447	5.1004	5.2473	5.3858	4.9341	3.4564
8	4.4243	4.6137	4.7900	4.9544	5.1079	5.2516	5.3864	4.9457	3.4653
9	4.4334	4.6243	4.8006	4.9636	5.1150	5.2557	5.3868	4.9569	3.4739
10	4.4424	4.6347	4.8109	4.9726	5.1217	5.2594	5.3869	4.9676	3.4823
11	4.4516	4.6452	4.8212	4.9816	5.1282	5.2629	5.3870	4.9782	3.4907
12	4.4607	4.6555	4.8312	4.9901	5.1344	5.2661	5.3868	4.9883	3.4988
13	4.4698	4.6658	4.8410	4.9984	5.1404	5.2692	5.3864	4.9981	3.5069
14	4.4790	4.6760	4.8506	5.0065	5.1462	5.2721	5.3859	5.0076	3.5147
15	4.4881	4.6861	4.8602	5.0144	5.1516	5.2746	5.3854	5.0168	3.5224
16	4.4973	4.6961	4.8696	5.0220	5.1569	5.2771	5.3847	5.0257	3.5299
17	4.5065	4.7060	4.8788	5.0294	5.1620	5.2793	5.3838	5.0343	3.5373
18	4.5157	4.7159	4.8877	5.0367	5.1668	5.2813	5.3829	5.0426	3.5445
19	4.5249	4.7257	4.8966	5.0436	5.1713	5.2832	5.3818	5.0506	3.5516
20	4.5341	4.7353	4.9053	5.0505	5.1758	5.2849	5.3807	5.0583	3.5585

zero coupon rates could cause a dish-shaped yield curve for coupon-bearing bonds. This finding is very disturbing, since it means that empirically observed changes in the shapes of yield curves for coupon-bearing bonds do not necessarily imply similar changes in the after-tax zero coupon yield curve, which represents market clearing rates.

Empirical studies have found that yield curves for coupon-bearing bonds tend to slope upward more often than not. The results in Table 2 indicate that, for the case of level, but shifting, after-tax zero coupon rates, yield curves for coupon bearing bonds can slope upward more often than not. Unless bonds selling at discounts from par are continuously replaced by par bonds, this type of upward bias in yield curves is a distinct possibility. Note that in practice premium bonds, and their downward sloping yield curves (see Table 2), are very often "called out of existence".

Since after-tax zero coupon discount rates are market equilibrating rates, the after-tax zero coupon forward rates are of considerable importance (See footnote #5 above). By assumption, the after-tax zero coupon forward rate implied by Table 2 is 4% for all maturities. The reader can easily verify that forward rates computed from the yields of coupon-bearing bonds in Table 2 (i.e. the f_j's in equation (5)) take on a wide variety of shapes for non-par bonds; these shapes will be very similar to the shapes of the yield curves in Table 2. An analyst looking exclusively at these f_j's might come to highly erroneous conclusions.

Table 2
Bond Yield Percentage
Constant After-Tax Zero Coupon Rate of 4%; TP = 30%; TG = 15%

	Coupon (%)								Before Tax-
Maturity	2	3	4	5	6	7	8	Coupon of Par Bond	Zero Coupon Rate
1	5.0694	5.2468	5.4214	5.5933	5.7623	5.9288	6.0927	5.7142	4.7058
2	5.0647	5.2454	5.4216	5.5937	5.7619	5.9261	6.0866	5.7142	4.6900
3	5.0606	5.2443	5.4222	5.5944	5.7614	5.9233	6.0804	5.7142	4.6746
4	5.0570	5.2436	5.4228	5.5952	5.7609	5.9205	6.0741	5.7142	4.6597
5	5.0539	5.2433	5.4239	5.5960	5.7604	5.9174	6.0676	5.7142	4.6454
6	5.0514	5.2435	5.4251	5.5970	5.7599	5.9144	6.0611	5.7142	4.6316
7	5.0493	5.2440	5.4265	5.5980	5.7593	5.9112	6.0545	5.7142	4.6183
8	5.0477	5.2449	5.4283	5.5992	5.7587	5.9080	6.0480	5.7142	4.6053
9	5.0464	5.2460	5.4301	5.6003	5.7581	5.9047	6.0413	5.7142	4.5927
10	5.0457	5.2475	5.4321	5.6015	5.7575	5.9014	6.0346	5.7142	4.5806
11	5.0454	5.2493	5.4344	5.6028	5.7568	5.8980	6.0280	5.7142	4.5688
12	5.0454	5.2514	5.4367	5.6041	5.7562	5.8947	6.0214	5.7142	4.5573
13	5.0458	5.2536	5.4391	5.6056	5.7556	5.8913	6.0148	5.7142	4.5463
14	5.0467	5.2563	5.4418	5.6070	5.7548	5.8879	6.0082	5.7142	4.5355
15	5.0478	5.2592	5.4446	5.6084	5.7541	5.8845	6.0016	5.7142	4.5251
16	5.0494	5.2622	5.4474	5.6099	5.7535	5.8811	5.9951	5.7142	4.5150
17	5.0513	5.2655	5.4505	5.6115	5.7538	5.8776	5.9887	5.7142	4.5052
18	5.0535	5.2690	5.4535	5.6130	5.7520	5.8742	5.9824	5.7142	4.4956
19	5.0560	5.2727	5.4568	5.6146	5.7514	5.8709	5.9761	5.7142	4.4864
20	5.0589	5.2767	5.4600	5.6162	5.7507	5.8674	5.9698	5:7142	4.4774

V

In addition to interest in the shape of yield curves, students of bonds are interested in making inferences about tax rates and after-tax zero coupon discount rates on the basis of observable yields to maturity of coupon-bearing bonds. The previous section presented some examples which showed the difficulty in using coupon-bearing bond yields to make inferences about market equilibrating after-tax zero coupon discount rates. The problem is further complicated by two factors. First, the shape of empirically observed yield curves can be influenced by the coupon levels of bonds. For example, it is easy to construct declining, rising, humpbacked, or dish-shaped curves by selecting appropriate coupon levels for particular maturities. Also, changes in the shape of the yield curve can result from the replacement of old non-par bond issues by new issues of par bonds. Secondly, the implicit marginal tax rate at different maturities does not have to be the same. Conceivably, the taxation of market participants at different maturities can create "segments" of the yield curve. Also, it is possible to have a level yield curve for par bonds with non-level TR_j, if tax rates differ between maturities.

In general, the yield to maturity y_j of a j period bond with coupon C is a function of the TR_j, TP_j, TG_j, where TP_j and TG_j are tax rates for j period bonds. That is, in general, there are $3N$ unknowns (i.e., N TR_j's, N TP_j's, N TG_j's). The question arises as to when inferences can be made about the shape of the TR_j

and tax rates; several cases are possible. (1) If one discount bond (i.e. selling below par) exists for each maturity, we have N equations and 3 N unknowns; no solution can be found. If TP and TG were constant for all maturities no solution can be found because there are $N + 2$ unknowns (i.e., TP, TG, and N TR_j's); if $TP = 2TG$ for all maturities, no solution is possible because there are $N + 1$ equations and $N + 2$ unknowns. (2) If two discount bonds are available for each maturity, the price of a taxable annuity and zero coupon bond can be derived. No solution can be found, since there are $2N$ equations (i.e. N equations for the price of zero coupon bonds and N equations for the prices of annuities) and 3 N unknowns. If $TP_j = 2TG_j$ for all j, there $3N$ unknowns and $3N$ equations, and a solution can be found. (3) In the special case of a flat yield curve for par bonds, the TR_j's are known to be flat (if the tax rates are the same for all maturities) but the level of the TR_j's cannot be determined.

These results are quite disturbing since they indicate that inferences about the shape (or levels) of market equilibrating after-tax zero coupon yield curves cannot be made from the shapes (or levels) of observable coupon-bearing bond yield curves. Consequently, existing empirical work testing theories of the behavior of zero coupon bonds, but employing coupon-bearing bond data, must be regarded with considerable reservation.

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