The Journal of Financial Research • Vol. XXVIII, No. 3 • Pages 343–361 • Fall 2005

EXPONENTIAL DURATION: A MORE ACCURATE ESTIMATION OF INTEREST RATE RISK

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Abstract

We develop a new method to estimate the interest rate risk of an asset. This method is based on modified duration and is always more accurate than traditional estimation with modified duration. The estimates by this method are close to estimates using traditional duration plus convexity when interest rates decrease. If interest rates rise, investors will suffer larger value declines than predicted by traditional duration plus convexity estimate. The new method avoids this undesirable value overestimation and provides an estimate slightly below the true value. For risk-averse investors, overestimation of value declines is more desirable and conservative.

JEL Classification: G10

I. Introduction

In 1938, Macaulay described what he called bond duration as a measure of average bond maturity. Ever since, there have been hundreds of articles and many books written about Macaulay's duration.[1] Macaulay's duration has often been used as a measure of the sensitivity of bond prices to changes in interest rates (or interest rate risk), as a tool in protecting bond portfolios from changing interest rates, and as an indicator of vulnerability of the equity value of financial institutions to changing interest rates.

A more commonly used measure of duration is modified Macaulay's duration, or modified duration. It equals the Macaulay's duration divided by 1 plus

We would like to thank Terry Nixon, David Shull, Donald Smith (the reviewer), and seminar participants at the 2003 Eastern Financial Association Conference in Orlando, Florida, for many insightful comments and suggestions that greatly improved this paper.

[1]Hawawini (1982) and Bierwag (1987) have extensive bibliographies. Hawawini also describes the development of the concept of duration and points out that other authors may have developed it independently, including Lidstone (1895), Hicks (1939), and Samuelson (1945). Duration is discussed by many authors, including Grove (1966, 1974) and Sundaresan (2002).

bond yield to maturity (Y).[2] The modified duration is defined as minus the first derivative of bond price with respect to Y divided by price and is a measure of the instantaneous percentage price sensitivity of a bond to changes in interest rate. For a small change in interest rates, modified duration provides a good approximation of the actual change in value. As the change in interest rates gets larger, the duration approximation has larger errors. The second derivative term, or convexity, is often used as a way to improve the accuracy of the approximation. Bond duration and convexity are widely used by practitioners, and the influential Bloomberg database reports these two measures of interest rate risk for each outstanding bond issue.

Duration is also widely used in the banking and financial industries to manage interest rate risk. As banks usually have short-term liabilities (time deposits, CDs, etc.) and longer term assets (long-term loans, mortgages, etc.), changes in interest rates may have a direct effect on the banks' equity value because of the mismatch of duration of their assets and liabilities. The gap between the durations of the assets and liabilities is a measure of the interest rate risk of banks' equity (e.g., see Koch 1995). Furthermore, large duration gaps reported by Fannie Mae and Freddie Mac have a significant effect on the Treasuries market as investors expect the two mortgage giants to buy and sell large quantities of Treasuries to bring the gap back to their target ranges.

Traditional modified duration estimation is a relatively good predictor of changes in the value of an asset for small interest rate changes but can be inaccurate for large interest rate changes. In addition, the traditional duration plus convexity method gives estimated values that are above actual values when interest rates increase. Consequently, investors may suffer larger value declines for rising interest rates than predicted by the traditional duration plus convexity method. In this article we analytically derive a simple, but accurate, estimate of the interest rate risk of fixed-income assets that avoids both of these problems.

We call our new estimate the exponential duration and analytically show it always to be more accurate than traditional estimation by modified duration, with dramatically greater accuracy for large interest rate changes. The reason for greater accuracy is the difference in estimation technique. The traditional duration method approximates the percentage change in value by approximating the absolute change in value and then dividing by initial value. The exponential duration approximation is based on the change in the natural logarithm of value, and this use of the natural logarithm is proved to give a more accurate approximation.

The absolute value of estimation errors using exponential duration is shown by numerical evaluation to be close to the absolute value of errors using the

[2]More precisely, the modified duration should equal the Macaulay duration divided by 1 plus bond yield per period to maturity when the frequency of coupon payment is more than once per year (see Smith 1998). However, for simplicity, we assume all bonds have annual coupon payment. We thank Don Smith for pointing this out.

traditional duration plus convexity method. In addition, for rising interest rates, the traditional duration plus convexity method overestimates asset value and understates the actual decline in asset value. Risk-averse investors should find this understatement of the risk of value decline undesirable. In contrast, the exponential duration estimate always lies below the true value and thus slightly overstates the value decline for rising interest rates—a desirable property for risk-averse investors. Thus, although the absolute errors for exponential duration are not always the same as traditional duration plus convexity, exponential duration is appealing for risk-averse investors.[3]

We use examples of several bonds with different maturities and coupon rates to illustrate the superiority of the new method. However, the new estimation technique can also be applied to duration gap (GAP) management by banks. Because banks usually have many different assets and liabilities, calculating their actual changes in value due to changes in interest rates is impractical and inefficient. Therefore, a simple and accurate estimation technique is desirable for banks to evaluate their interest rate risk.

The value at risk (VAR) approach (see Hopper 1996; Saunders and Allen 2002; RiskMetrics Group 2004) is widely used to measure the downside portfolio risk of financial institutions. The basic approach is to estimate the likely change in interest rates for a particular level of confidence and multiply this by the interest rate sensitivity of the portfolio, typically modified duration. A more accurate measure of interest rate sensitivity as developed here will increase the accuracy of the VAR approach.

Also, for portfolio managers of fixed-income mutual funds, finding the actual changes in prices for the many securities in their portfolio is cumbersome. Use of exponential duration would give fund managers a simple and highly accurate estimate of bond price sensitivity to interest rate changes for their entire portfolio, providing the managers a quick and precise understanding of their portfolio's interest rate risk.

II. Traditional Approximation

For a flat term structure, the price of a bond or fixed-income asset with maturity of T may be expressed as the present value of coupons (C) and par value (Par) discounted at the yield to maturity (Y) as follows:

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[3] We also examined another estimation procedure that we call the exponential duration plus convexity method. The details are available from the authors. This technique gives more accurate estimates than estimation by traditional duration plus convexity but suffers from the same deficiency as the traditional duration plus convexity method. Namely, when interest rates increase, it typically overstates the value and understates the value decline.

$$P = \sum_{t=1}^{T} \frac{C}{(1+Y)^t} + \frac{Par}{(1+Y)^T}. \tag{1}$$

In the following discussion, we use the term "bond price" for simplicity, but the results hold for all fixed-income assets. If we take the first derivative of the price (P) with respect to yield to maturity (Y) and divide it by P, we get minus modified duration (D),[4] or:

$$D \equiv -\left(\frac{1}{P}\right)\frac{dP}{dY} = -\frac{dP/P}{dY}. \tag{2}$$

Modified duration provides an approximation of the percentage change in bond price in response to a change in interest rate. The modified duration estimates the percentage price changes as follows:

$$\frac{\Delta P}{P_0} \approx \frac{\tilde{P}_1 - P_0}{P_0} = -D \times \Delta Y, \tag{3}$$

where P_0 is the original price, $\tilde{P}_1$ is estimated new price, and ΔY is the change in interest rates. After some manipulation, the estimated price ($\tilde{P}_1$) is:

$$\tilde{P}_1 = P_0(1 - D \times \Delta Y). \tag{4}$$

Tables 1, 2, and 3 give some numerical examples of price estimates by the traditional duration method for three different bonds: a par bond, a zero-coupon bond, and a perpetual bond. For example, Table 1 assumes a 30-year par bond with a 5% annual coupon rate. The modified duration of the bond (D) is 15.37. Table 1 shows the actual new price (column 3) and estimated new price by the traditional modified duration method (column 4). For an increase of 50 basis points in the interest rate, the estimated new price can be found from equation (4) to be \$92.31, which is close to the actual new price of \$92.73.

However, the traditional duration method of estimation is not accurate when interest rate changes are large. In Table 1, for example, if the interest rate increases by 3%, the estimated new price is \$53.88, far from the actual new price of \$66.23.

[4]The modified duration has a closed-end solution as follows:

$$D = \frac{\left\{\frac{C}{Y^2}\left[1-(1+Y)^{-N}\right] + \frac{N(Par - C/Y)}{(1+Y)^{N+1}}\right\}}{P},$$

where C is the coupon payment, N is the maturity, and Par is the par value.

TABLE 1. Thirty-Year Par Bond with 5% Coupon Rate.

Changes in YTM	New YTM	Actual New Price	Traditional Duration Estimate	Traditional Duration & Convexity Estimate	Exponential Duration Estimate
−3.00%	2.00%	$167.19	$146.12	$161.89	$158.59
−2.50%	2.50%	$152.33	$138.43	$149.38	$146.86
−2.00%	3.00%	$139.20	$130.74	$137.75	$136.00
−1.50%	3.50%	$127.59	$123.06	$127.00	$125.93
−1.00%	4.00%	$117.29	$115.37	$117.12	$116.62
−0.50%	4.50%	$108.14	$107.69	$108.12	$107.99
0.00%	5.00%	$100.00	$100.00	$100.00	$100.00
0.50%	5.50%	$92.73	$92.31	$92.75	$92.60
1.00%	6.00%	$86.24	$84.63	$86.38	$85.75
1.50%	6.50%	$80.41	$76.94	$80.88	$79.41
2.00%	7.00%	$75.18	$69.26	$76.26	$73.53
2.50%	7.50%	$70.47	$61.57	$72.52	$68.09
3.00%	8.00%	$66.23	$53.88	$69.65	$63.05

Note: This table gives the price sensitivity of a 30-year par bond with 5% annual coupon rate. The modified duration (D) is 15.37. The convexity (V) is 175.23. The third column gives the actual new price of the bond when yield to maturity (YTM) changes. The fourth column gives the estimated new price by the traditional duration method ($\tilde{P}_1 = P_0(1 - D \times \Delta Y)$). The fifth column gives the estimated new price by the traditional duration plus convexity method ($\tilde{P}'_1 = P_0(1 - D \times \Delta Y + V \times \Delta Y^2)$). The sixth column gives the estimated new price by the exponential duration method ($\tilde{\tilde{P}}_1 = P_0 \times e^{-D\times\Delta Y}$).

TABLE 2. Thirty-Year Zero-Coupon Bond.

Changes in YTM	New YTM	Actual New Price	Traditional Duration Estimate	Traditional Duration & Convexity Estimate	Exponential Duration Estimate
−3.00%	2.00%	$55.21	$42.97	$51.75	$54.52
−2.50%	2.50%	$47.67	$39.66	$45.76	$47.26
−2.00%	3.00%	$41.20	$36.36	$40.26	$40.97
−1.50%	3.50%	$35.63	$33.05	$35.25	$35.52
−1.00%	4.00%	$30.83	$29.75	$30.72	$30.79
−0.50%	4.50%	$26.70	$26.44	$26.69	$26.69
0.00%	5.00%	$23.14	$23.14	$23.14	$23.14
0.50%	5.50%	$20.06	$19.83	$20.08	$20.06
1.00%	6.00%	$17.41	$16.53	$17.50	$17.39
1.50%	6.50%	$15.12	$13.22	$15.42	$15.07
2.00%	7.00%	$13.14	$9.92	$13.82	$13.07
2.50%	7.50%	$11.42	$6.61	$12.71	$11.33
3.00%	8.00%	$9.94	$3.31	$12.09	$9.82

Note: This table gives the price sensitivity of a 30-year zero coupon bond. The modified duration is 28.57 (D). The convexity (V) is 421.77. The third column gives the actual new price of the bond when yield to maturity (YTM) changes. The fourth column gives the estimated new price by the traditional duration method ($\tilde{P}_1 = P_0(1 - D \times \Delta Y)$). The fifth column gives the estimated new price by the traditional duration plus convexity method ($\tilde{P}'_1 = P_0(1 - D \times \Delta Y + V \times \Delta Y^2)$). The sixth column gives the estimated new price by the exponential duration method ($\tilde{\tilde{P}}_1 = P_0 \times e^{-D\times\Delta Y}$).

TABLE 3. Perpetuity with 5% Coupon Rate.

Changes in YTM	New YTM	Actual New Price	Traditional Duration Estimate	Traditional Duration & Convexity Estimate	Exponential Duration Estimate
−3.00%	2.00%	$250.00	$160.00	$196.00	$182.21
−2.50%	2.50%	$200.00	$150.00	$175.00	$164.87
−2.00%	3.00%	$166.67	$140.00	$156.00	$149.18
−1.50%	3.50%	$142.86	$130.00	$139.00	$134.99
−1.00%	4.00%	$125.00	$120.00	$124.00	$122.14
−0.50%	4.50%	$111.11	$110.00	$111.00	$110.52
0.00%	5.00%	$100.00	$100.00	$100.00	$100.00
0.50%	5.50%	$90.91	$90.00	$91.00	$90.48
1.00%	6.00%	$83.33	$80.00	$84.00	$81.87
1.50%	6.50%	$76.92	$70.00	$79.00	$74.08
2.00%	7.00%	$71.43	$60.00	$76.00	$67.03
2.50%	7.50%	$66.67	$50.00	$75.00	$60.65
3.00%	8.00%	$62.50	$40.00	$76.00	$54.88

Note: This table gives the price sensitivity of a perpetuity bond with 5% annual coupon rate. The modified duration (D) is 20. The convexity (V) is 400. The third column gives the actual new price of the bond when yield to maturity (YTM) changes. The fourth column gives the estimated new price by the traditional duration method ($\tilde{P}_1 = P_0(1 - D \times \Delta Y)$). The fifth column gives the estimated new price by the traditional duration plus convexity method ($\tilde{P}'_1 = P_0(1 - D \times \Delta Y + V \times \Delta Y^2)$). The sixth column gives the estimated new price by the exponential duration method ($\tilde{\tilde{P}}_1 = P_0 \times e^{-D\times\Delta Y}$).

To achieve a better approximation, financial practitioners often adjust the traditional duration approximation with convexity as follows:

$$\tilde{P}'_1 = P_0(1 - D \times \Delta Y + V \times \Delta Y^2), \tag{5}$$

where $V \equiv \frac{1}{2P_0} \times \frac{d^2P}{dY^2}$ and is often called convexity.[5] $\tilde{P}'_1$ is the estimated new price by the traditional duration method with a convexity correction. V, or convexity, is simply the second derivative of the bond price divided by $2P_0$.

Correction with convexity has been shown to yield smaller estimation errors, especially for large interest rate changes. Column 5 in Tables 1, 2, and 3 gives price estimates by the traditional duration plus convexity method. For the 30-year, 5% par bond, the convexity (V) is 175.23. When the interest rate increases by 3%, the estimated new price by the traditional duration plus convexity method is $69.65, much closer to the actual new price of $66.23 than the estimates with duration only.

[5]The convexity, V, has a closed-end solution as follows:

$$V = \frac{\frac{2C}{Y^3}[1 - (1+Y)^{-N}] - \frac{2C*N}{Y^2(1+Y)^{N+1}} + \frac{N(N+1)(Par - C/Y)}{(1+Y)^{N+2}}}{2P}.$$

However, for interest rate increases, using traditional duration plus convexity overstates the price compared with the actual price as is seen in Tables 1, 2, and 3 by comparing column 5 with column 3. For risk-averse investors this overstatement is a problem because their positions may suffer larger price declines for rising interest rates than predicted by the traditional duration plus convexity method. Thus, the traditional duration plus convexity method understates the risk of price decline for rising interest rates.

III. A New Approximation: Exponential Duration Method

Several authors point out that the natural logarithm of bond price is a better measure of percentage changes in bond prices as interest rates change (see Bierwag, Kaufman, and Latta 1988; Campbell, Lo, and MacKinlay 1997; Crack and Nawalkha 2001). Based on this idea, we derive a simple but highly accurate method of estimating percentage bond price changes in response to changes in interest rates. If we take the derivative of the natural log of price ($\ln P$) with respect to yield to maturity (Y), we get minus modified duration (D).

$$\frac{d(\ln P)}{dY} = \left(\frac{1}{P}\right)\left(\frac{dP}{dY}\right) = -D. \tag{6}$$

Rearranging, we get:

$$d(\ln P) = -D \times dY. \tag{7}$$

If dY is approximated by ΔY, then

$$d(\ln P) = -D \times \Delta Y. \tag{8}$$

Suppose that we can approximate $d(\ln P)$ by $\Delta(\ln P)$. Then

$$\Delta(\ln P) \approx \ln \tilde{\tilde{P}}_1 - \ln P_0, \tag{9}$$

where P_0 is the original price and $\tilde{\tilde{P}}_1$ is the approximated new price following a change in interest rate.

Equation (9) uses the change in the natural logarithm of bond price as an approximation of the percentage change in bond price. The traditional modified duration is another measure of the percentage change in bond price that approximates the change in bond price and then divides by initial price. Either approximation is conceptually feasible, but the better approximation is the more accurate one. Next, using the change in natural logarithms, equation (9) is shown analytically to always be a more accurate approximation.

It follows from equation (9) that

$$\ln \tilde{\tilde{P}}_1 - \ln P_0 = -D \times \Delta Y. \tag{10}$$

This expression can be rewritten as

$$\ln\left(\frac{\tilde{\tilde{P}}_1}{P_0}\right) = -D \times \Delta Y. \tag{11}$$

The new price can be approximated by

$$\tilde{\tilde{P}}_1 = P_0 \times e^{-D\times\Delta Y}, \tag{12}$$

and the percentage change in price can be approximated by

$$\%\Delta P \approx (e^{-D\times\Delta Y} - 1) * 100. \tag{13}$$

Because this approximation has the modified duration (*D*) in the exponent, we call it the exponential duration method.

IV. The Greater Accuracy of the Exponential Duration Method

This new exponential duration method yields more accurate price estimates than the traditional duration method. To see this, we can rewrite the new exponential duration approximation in equation (12) as follows by expanding $e^{-D\times\Delta Y}$ in a Taylor series expansion of the exponential function:

$$\begin{aligned}\tilde{\tilde{P}}_1 &= P_0\Bigg(1 + (-D\times\Delta Y) + \frac{(-D\times\Delta Y)^2}{2!} + \frac{(-D\times\Delta Y)^3}{3!} \\ &\quad + \cdots + \frac{(-D\times\Delta Y)^n}{n!} + \cdots\Bigg) \\ &= P_0 \sum_{n=0}^{\infty}\left(\frac{(-D\times\Delta Y)^n}{n!}\right).\end{aligned} \tag{14}$$

Thus, exponential duration approximation includes the traditional duration approximation as well as some additional terms. The exponential duration estimate is always larger than the traditional duration estimate. To see this, subtract the right-hand side of equation (4) from the right-hand side of equation (14). The additional terms in equation (14) always make the exponential duration approximation

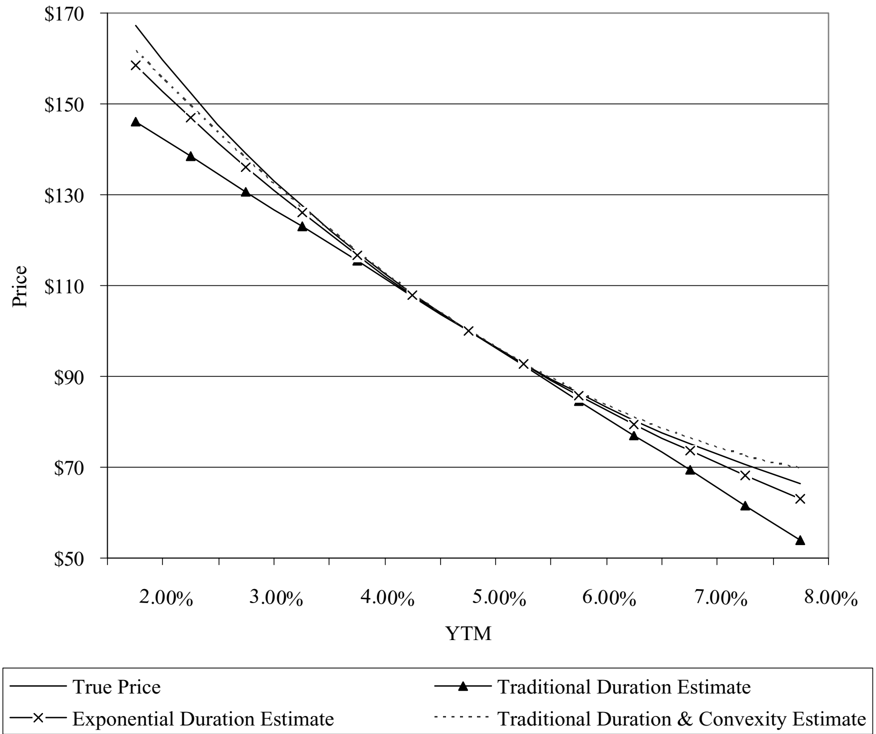

Figure I. Price-Yield Relation for a 30-Year, 5% Par Bond. This figure assumes a 30-year, 5% par bond. It plots the true new price and price estimates by three estimation methods—traditional duration, exponential duration, and traditional duration plus convexity—when the yield to maturity (YTM) changes by 3%.

a more accurate approximation than the traditional duration approximation. This is formally proved in the Appendix.

The basic intuition behind the greater accuracy of exponential duration is illustrated in Figure I. The traditional duration approximation is a straight-line function, whereas the exponential duration approximation has curvature. For traditional duration, interest rate increases and decreases have the same effect on the approximate changes in bond price. In contrast, the exponential duration approximation has a different effect on the approximate price change for interest rate increases versus interest rate decreases. In mathematical terms, the exponential approximation has e to a power, namely, minus modified duration times the change in the interest rate ($e^{-D\times\Delta Y}$) minus 1 (see equation (13)). When the interest rate increases, the exponent of the first term of equation (13) is negative and the first term of the approximation ($e^{-D\times\Delta Y}$) is less than 1. When the interest rate decreases, the exponent of the first term ($e^{-D\times\Delta Y}$) of equation (13) is positive and the first term in the approximation is greater than 1.

The improved accuracy of the new exponential duration method can be illustrated by the examples in Tables 1, 2, and 3. The sixth column gives the exponential duration estimates.

For the 30-year, 5% par bond in Table 1, when the interest rate increases by 50 basis points, the exponential duration method gives a new price of $92.60, close to the actual new price of $92.73 and better than the traditional duration estimate of $92.31. The superiority of the new method is more obvious when the interest rate change is large. When the interest rate increases by 3%, the exponential duration method gives an estimated new price of $63.05 versus the actual new price of $66.23. This is a significantly better estimation than the approximation of $53.88 by the traditional duration method. Figure I further illustrates the superiority of the exponential duration estimate by plotting the true price and three different price estimates for the 30-year, 5% par bond when the interest rate changes by 3%.

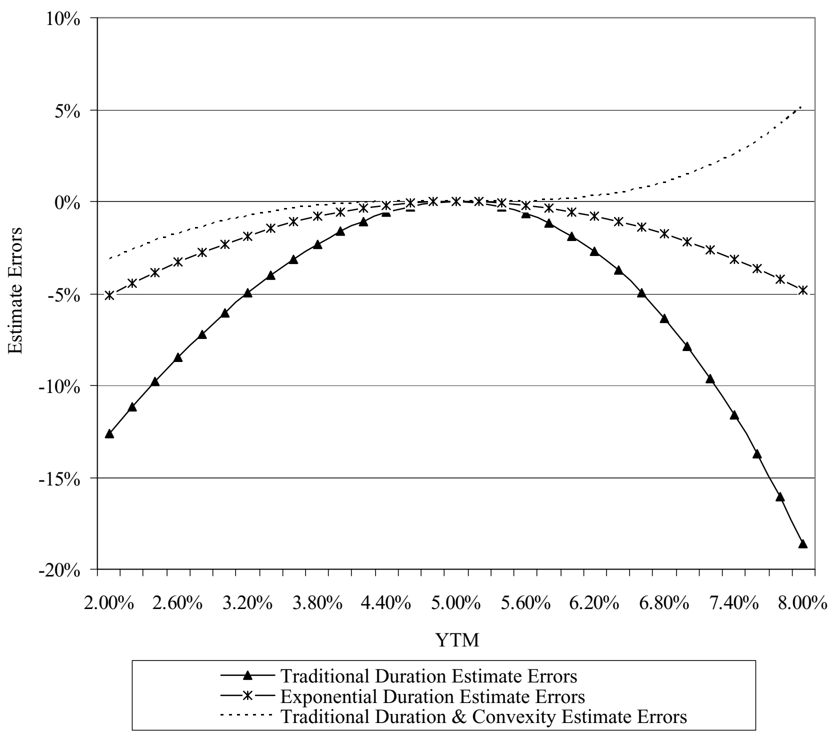

Figure II. Estimation Errors for a 30-Year, 5% Par Bond. This figure assumes a 30-year, 5% par bond. It plots the estimation errors as a percentage of true prices for three estimation methods—traditional duration, exponential duration, and traditional duration plus convexity—when the yield to maturity (YTM) changes by 3%.

It is obvious from these examples that the exponential duration method performs better than the traditional duration method, especially when interest rate changes are fairly large. Although the exponential duration method does not give the same estimate as the traditional duration plus convexity method, it is very close to the actual new price, even when interest rate changes are large. This issue is covered in more detail in the next section.

Table 2 gives an example of a 30-year, zero-coupon bond, and Table 3 gives an example of a perpetuity bond with 5% coupon rate. In the case of the zero-coupon bond, the exponential duration method gives the most accurate price estimation among the three methods.

To illustrate better the superiority of the exponential duration estimates, we plot the estimation errors as a percentage of the true prices for the three examples in the previous tables in Figures II through IV. The exponential duration estimate has far smaller estimation errors than the traditional duration estimate, regardless of the bond maturity and coupon rate.

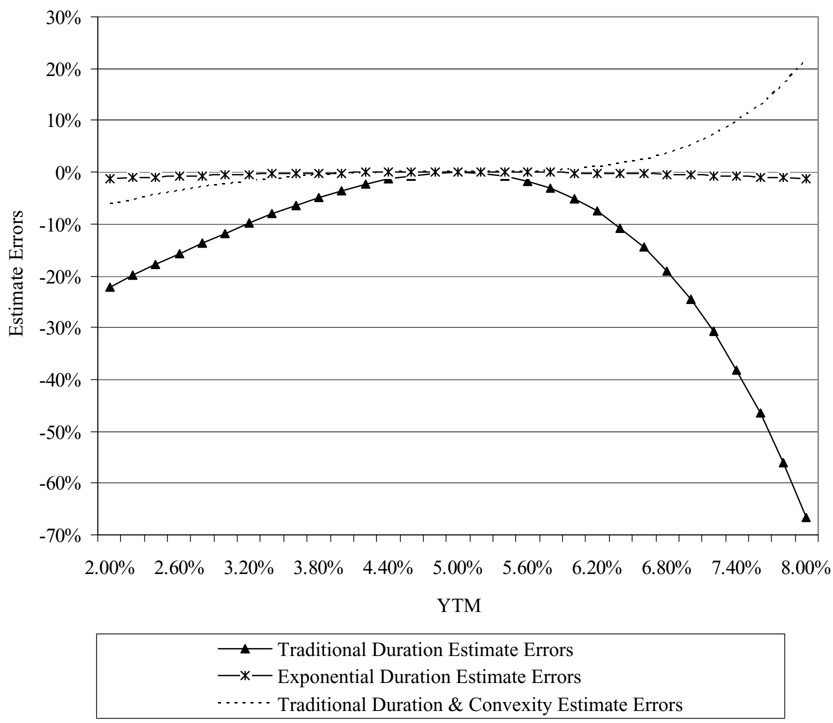

Figure III. Estimation Errors for a 30-Year, Zero-Coupon Bond. This figure assumes a 30-year, zero-coupon bond with 5% yield to maturity (YTM). It plots the estimation errors as a percentage of true prices for three estimation methods—traditional duration, exponential duration, and traditional duration plus convexity—when the YTM changes by 3%.

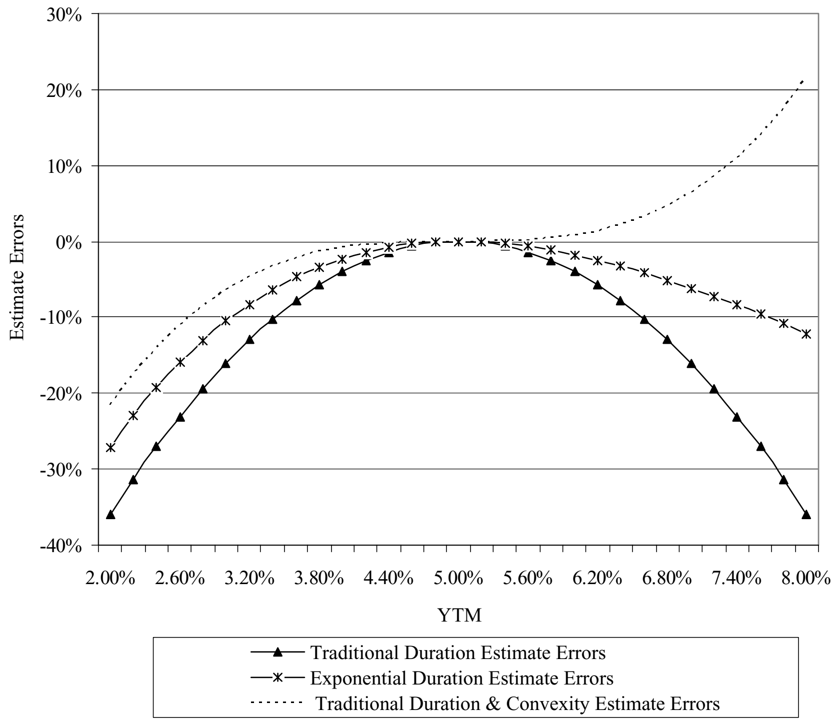

Figure IV. Estimation Errors for a 5% Perpetuity Bond. This figure assumes a 5% perpetual bond. It plots the estimation errors as a percentage of true prices for three estimation methods—traditional duration, exponential duration, and traditional duration plus convexity—when the yield to maturity (YTM) changes by 3%.

V. Exponential Duration Versus Traditional Duration Plus Convexity Method

As shown in section II and the previous numerical examples, the traditional duration method can be improved by adding the second term of a Taylor expansion of bond price—so-called bond convexity—into the estimation. This second derivative term reduces the absolute value of the estimation errors, and the traditional duration plus convexity method gives more accurate estimates compared with the traditional duration method. However, for increases in interest rates, traditional duration plus convexity results in an estimate of bond price that lies above the true bond price. This overestimation is simply due to the fact that the third derivative of bond price with respect to yield to maturity is negative.

The overestimation of bond price by the traditional duration plus convexity method is undesirable for risk-averse investors. These risk-averse investors are more concerned with the adverse effect of price declines from rising interest rates than with the favorable benefits of price increases with falling interest rates. That is, price declines are more undesirable than price increases of the same magnitude. On the other hand, exponential duration estimates always lie below the true price because the second derivative of the natural logarithm of price with respect to yield to maturity is positive.

Though the traditional duration plus convexity method gives smaller errors than exponential duration estimation in most cases, the difference is small. Figure II compares the estimation errors for a 30-year 5% par bond for traditional duration, exponential duration, and traditional duration plus convexity. Exponential duration is considerably more accurate than traditional duration. Exponential duration has slightly larger errors compared with traditional duration plus convexity for falling interest rates, and about the same absolute level of error for rising interest rates, with the added advantage that exponential duration provides a more conservative estimate of the price decline for rising interest rates.

Figure III shows the estimation errors for a 30-year, zero-coupon bond. The exponential duration has remarkably low estimation errors compared with traditional duration plus convexity and traditional deviation methods. Figure IV presents estimation errors for a perpetual bond. As in Figure II, exponential duration is considerably more accurate than traditional duration. For declining interest rates,

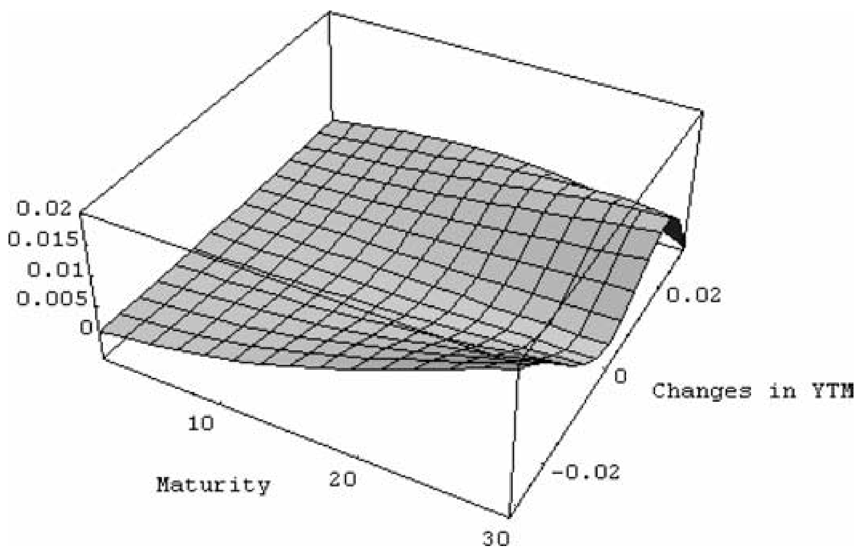

Figure V. Comparison of Exponential Duration Estimation Errors and Traditional Duration Plus Convexity Estimation Errors for 5% Par Bonds. This figure assumes 5% par bonds. The Maturity axis varies the maturity of the bond from 1 year to 30 years, and the Changes in Yield to Maturity (YTM) axis varies changes in YTM from −3% to 3%. The vertical axis gives the difference in absolute estimation errors between the exponential duration and the traditional duration plus convexity method and as a fraction of true prices. If the difference shown on the vertical axis is positive (negative), the exponential duration estimates have larger (smaller) absolute errors. Thus, a reading of 0.02 on the vertical axis shows that the exponential duration estimate error is 2% of the true price larger than the traditional duration plus convexity estimate.

exponential duration has larger errors than traditional duration plus convexity, and for increasing interest rates, exponential duration has approximately the same errors, but the estimated prices are lower than the actual price.

After doing an extensive numerical comparison of the absolute value of estimation errors as a percentage of true prices for the exponential duration versus the traditional duration plus convexity methods, we find that the difference is almost always less than 2% of true prices even when the interest rate changes by 3%. In Figures V, VI, and VII, we plot the difference in absolute estimation errors as a percentage of true prices for three bonds—a 5% par bond, a 10% par bond, and a zero-coupon bond of 5% yield to maturity—against different maturities and changes in interest rate. If the difference is positive (negative), the exponential duration estimates have larger (smaller) absolute errors.

Figures V and VI compare exponential duration errors with the errors from traditional duration plus convexity for maturities up to 30 years assuming 5% par bonds and 10% par bonds. In both figures, the errors for exponential duration are only slightly larger, and all are small for shorter maturities.

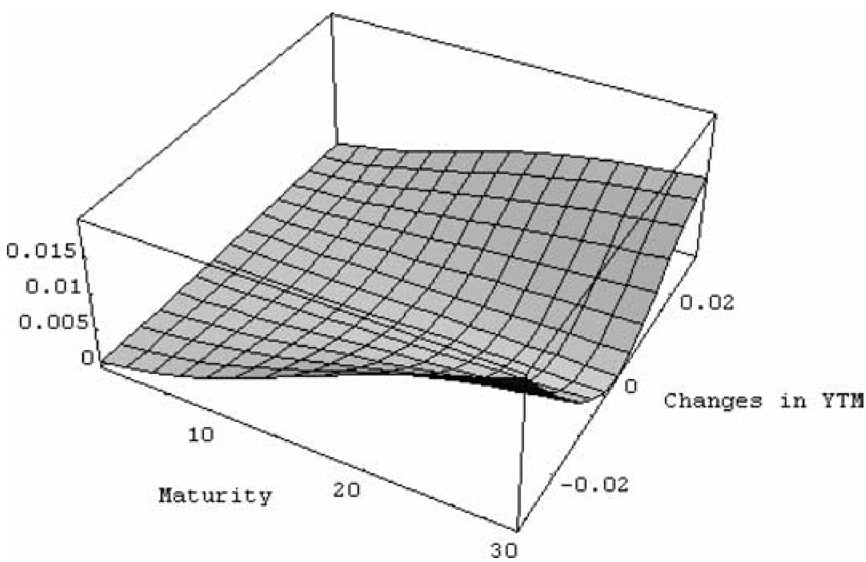

Figure VI. Comparison of Exponential Duration Estimation Errors and Traditional Duration Plus Convexity Estimation Errors for 10% Par Bonds. This figure assumes 10% par bonds. The Maturity axis varies the maturity of the bond from 1 year to 30 years, and the Changes in Yield to Maturity (YTM) axis varies changes in YTM from −3% to 3%. The vertical axis gives the difference in absolute estimation errors between the exponential duration and the traditional duration plus convexity method and as a fraction of true prices. If the difference shown on the vertical axis is positive (negative), the exponential duration estimates have larger (smaller) absolute errors. Thus, a reading of 0.015 on the vertical axis shows that the exponential duration estimate error is 1.15% of the true price larger than the traditional duration plus convexity estimate.

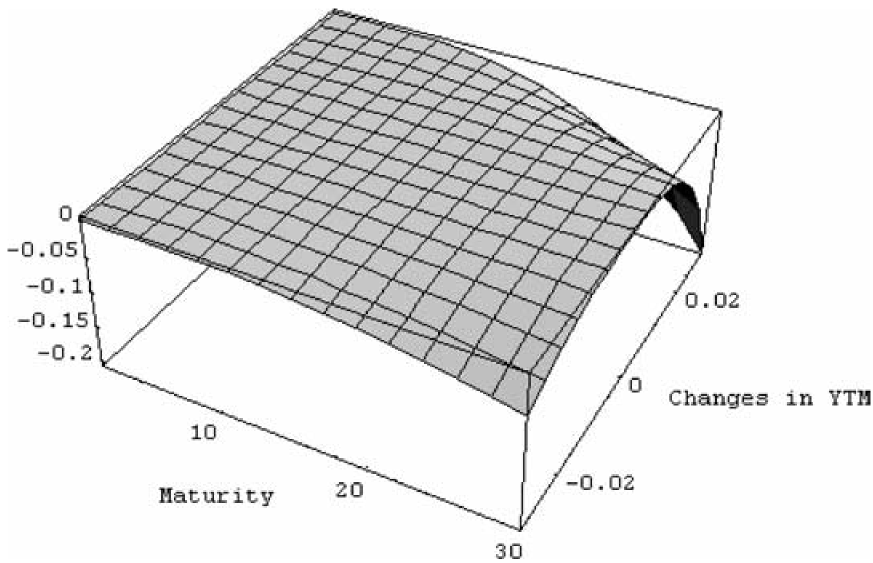

Figure VII. Comparison of Exponential Duration Estimation Errors and Traditional Duration Plus Convexity Estimation Errors for Zero-Coupon Bonds with 5% Yield to Maturity (YTM). This figure assumes zero-coupon bonds with 5% YTM. The Maturity axis varies the maturity of the bond from 1 year to 30 years, and the Changes in YTM axis varies changes in YTM from −3% to 3%. The vertical axis gives the difference in absolute estimation errors between the exponential duration and the traditional duration plus convexity method and as a fraction of true prices. If the difference shown on the vertical axis is positive (negative), the exponential duration estimates have larger (smaller) absolute errors. Thus, a reading of −0.05 on the vertical axis shows that the exponential duration estimate error is 5% of the true price smaller than the traditional duration plus convexity estimate.

Figure VII examines the errors for the zero-coupon bonds with a 5% yield to maturity for maturities up to 30 years. The exponential duration is always more accurate than traditional duration plus convexity (because the error is negative in sign), with considerably more accuracy for long maturities and interest rate increases. In summary, these figures clearly indicate that the absolute value of errors from exponential duration are remarkably close to the absolute value of errors from the estimation using traditional duration plus convexity.[6] When the interest rate decreases, the exponential duration estimates are almost as accurate as the traditional duration plus convexity estimates. For rising interest rates, exponential duration estimates have approximately the same absolute value of errors but do not understate the decrease in value.

[6]We have similar figures for par bonds with coupon rates from 1% to 15%. For all these bonds, the differences in errors are less than 2% of the true prices. These figures are available on request.

VI. Conclusion

We develop a new method called exponential duration to estimate the interest rate risk of an asset. This method has several desirable properties. First, exponential duration is much more accurate than traditional estimation with modified duration and is almost as accurate as estimation with traditional modified duration plus convexity. This allows fund managers, banks, and other businesses to have a more accurate estimate of the interest rate risk of their asset without finding the actual changes in values of many assets and liabilities in their portfolios. Second, exponential duration avoids computation of the traditional convexity measure because exponential duration is computed by simply raising the natural number, *e,* to a power. Finally, exponential duration slightly overestimates bond price declines when the interest rate increases. Risk-averse investors should find this desirable. In contrast, traditional duration plus convexity underestimates bond price declines for increasing interest rates. For these reasons, we believe this new exponential duration estimation technique is superior to the existing methods and should be widely adopted.

Appendix

In this appendix we prove that exponential duration is a more accurate approximation of bond price change than traditional modified duration.

First, let

P_1 = new price;
P_0 = old price;
ΔP = price change = $P_1 - P_0$;
Y_1 = new yield to maturity;
Y_0 = old yield to maturity;
ΔY = change in yield to maturity = $Y_1 - Y_0$; and
D = modified duration = $-\dfrac{dP}{dY}\Big/ P_0$.

By Taylor expansion, we can find the true price change in response to change in yield to maturity as follows:

$$\begin{aligned}\Delta P &= \frac{dP}{dY} \times \Delta Y + \frac{d^2P}{dY^2} \times \frac{\Delta Y^2}{2!} + \frac{d^3P}{dY^3} \times \frac{\Delta Y^3}{3!} + \cdots + \frac{d^nP}{dY^n} \times \frac{\Delta Y^n}{n!} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{d^nP}{dY^n} \times \frac{\Delta Y^n}{n!} \\ &= \left[\sum_{n=1}^{\infty} \frac{d^nP}{dY^n} \times \frac{\Delta Y^n}{n!} \Big/ P_0\right] \times P_0.\end{aligned}$$

Therefore,

$$
\begin{aligned}
P_1 &= P_0 + \Delta P \\
&= P_0 + \left[\sum_{n=1}^{\infty} \frac{d^n P}{dY^n} \times \frac{\Delta Y^n}{n!} \Big/ P_0\right] \times P_0 \\
&= P_0\left[1 + \sum_{n=1}^{\infty} \frac{d^n P}{dY^n} \times \frac{\Delta Y^n}{n!} \Big/ P_0\right] \\
&= P_0\left[1 - D \times \Delta Y + \sum_{n=2}^{\infty} \frac{d^n P}{dY^n} \times \frac{\Delta Y^n}{n!} \Big/ P_0\right]. \qquad \text{(A1)}
\end{aligned}
$$

Equation (4) gives $\tilde{P}_1$, the estimated new price by the traditional method, as follows:

$$\tilde{P}_1 = P_0(1 - D \times \Delta Y). \qquad \text{(A2)}$$

Let $\tilde{e}$ be the estimate error of the traditional method. Therefore,

$$
\begin{aligned}
\tilde{e} &= P_1 - \tilde{P}_1 \\
&= P_0 \times \sum_{n=2}^{\infty} \frac{d^n P}{dY^n} \times \frac{\Delta Y^n}{n!} \Big/ P_0. \qquad \text{(A3)}
\end{aligned}
$$

Equation (14) gives $\tilde{\tilde{P}}_1$, the estimated new price by the exponential duration method, as follows:

$$
\begin{aligned}
\tilde{\tilde{P}}_1 &= P_0\left[1 + (-D \times \Delta Y) + \frac{(-D \times \Delta Y)^2}{2!} + \frac{(-D \times \Delta Y)^3}{3!}\right. \\
&\quad \left. + \cdots + \frac{(-D \times \Delta Y)^n}{n!} + \cdots\right] \\
&= P_0\left[1 - D \times \Delta Y + \sum_{n=2}^{\infty} \frac{(-D \times \Delta Y)^n}{n!}\right]. \qquad \text{(A4)}
\end{aligned}
$$

Let $\tilde{\tilde{e}}$ be the estimate error of the exponential duration method. Therefore,

$$
\begin{aligned}
\tilde{\tilde{e}} &= P_1 - \tilde{\tilde{P}}_1 \\
&= P_0 \times \sum_{n=2}^{\infty} \left[\frac{d^n P}{dY^n} \times \frac{\Delta Y^n}{n!} \Big/ P_0 - \frac{(-D \times \Delta Y)^n}{n!}\right]. \qquad \text{(A5)}
\end{aligned}
$$

Now we have two estimation errors, one from the traditional modified duration estimation and the other from the exponential duration method. Note that the two error terms are both positive because the second derivatives of bond price, P, and log of price, $\ln P$, with respect to yield to maturity, or Y, are positive.[7] If we can prove that the estimation error from the traditional duration method is always greater than errors from the exponential duration method, the exponential duration method yields a better estimation. Therefore, let's compare the two estimate errors.

$$\begin{aligned}
\tilde{e} - \tilde{\tilde{e}} &= P_0 \times \sum_{n=2}^{\infty}\left[\frac{d^nP}{dY^n} \times \frac{\Delta Y^n}{n!} \Big/ P_0\right] - P_0 \\
&\quad \times \sum_{n=2}^{\infty}\left[\frac{d^nP}{dY^n} \times \frac{\Delta Y^n}{n!} \Big/ P_0 - \frac{(-D \times \Delta Y)^n}{n!}\right] \\
&= P_0 \times \sum_{n=2}^{\infty}\left[\frac{(-D \times \Delta Y)^n}{n!}\right] \\
&= P_0 \times \left\{\sum_{n=0}^{\infty}\left[\frac{(-D \times \Delta Y)^n}{n!}\right] - [1 + (-D \times \Delta Y)]\right\} \\
&= P_0 \times \left\{e^{(-D \times \Delta Y)} - [1 + (-D \times \Delta Y)]\right\}. \qquad \text{(A6)}
\end{aligned}$$

Let $D \times \Delta Y$ be x and we can rewrite the terms in the bracket of equation (A6) as:

$$z = e^{-x} - 1 + x.$$

Because $dz/dx = -e^{-x} + 1 = 0$ when $x = 0$ and $d^2z/dx = e^{-x} > 0$ for all x, z is minimized when $x = 0$. Also, because $z = 0$ when $x = 0$, it holds that $z \geq 0$. Hence, it is always true that

$$\tilde{e} - \tilde{\tilde{e}} = P_0 \times \left\{e^{(-D \times \Delta Y)} - [1 + (-D \times \Delta Y)]\right\} \geq 0 \quad \text{for all } D, \Delta Y, \text{ and } P_0.$$

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[7] It is straightforward to prove that the second derivative of bond price (P) with respect to yield to maturity (Y) is positive. The proof that the second derivative of the log of bond price ($\ln P$) with respect to yield to maturity (Y) is positive is fairly complicated. To conserve space, we omit the proof from the article. Interested readers can obtain the formal proof from the authors.

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